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A Snapshot of Advances in Auctions Research: Introduction to the Special Issue on Auctions of the **Applied Economics Research Bulletin**[*]

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Auctions have been used in one form or another since ancient times. Early examples frequently cited in the academic literature include the auctioning of brides by the Babylonians around 500 BCE as well as the use of auctions in ancient Rome for the likes of war booty, slaves, and debtors' properties—the entire Roman Empire was sold auction by the Praetorian Guard after the murder of the Emperor Pertinax in 193 CE. Many wholesale agricultural and fisheries markets also have long histories of making use of di□erent types of auctions. Auctions are also used to sell rare collectibles in prestigious auction houses; common household goods in estate auctions; financial instruments, electricity, and confiscated contraband in government-sponsored auctions; and nearly anything on the Internet at auction host sites which specialize in facilitating person-to-person, business-to-person or business-to-business transactions. Indeed, it is hard to identify goods or services that have not been traded at auction.

In the past few decades, the use of auctions has proliferated in two areas in particular. First, the aforementioned flourishing use of the Internet to facilitate or even to enable trade by establishing markets for all sorts of goods and services. And, second, auctions as a frequently preferred means for trades involving governments. Thus, much of the privatization waves in the aftermath of the demise of the Soviet Union as well as many other government divestitures, often tied to deregulation, have used auctions. Similarly, much of government procurement—from the purchase of o□ce furniture in small municipalities, to the construc-

[*] **The papers included in the special issue are available at the** *Applied Economics Research Bulletin***'s website at http://berkeleymath.com/BerkeleyJournal.aspx**

tion and maintenance of public highways at the county/regional/state level, to the purchase of fighter jets at the national level—is now conducted through some form of auction. Most notable, however, is the recent use of auctions as a means of allocating spectrum rights to private companies. The ubiquitousness of auctions has naturally led to a growth in academic research on auctions. However, this is surely not a one-way street. Indeed, as a case in point, the rather spectacular spectrum auctions conducted in many countries which have raised billions of dollars in revenue are the direct result of implementing new research insights gained in the design and operation of auctions.[1]

The study of auctions by academic economists may be traced back to Vickrey's (1961) now-seminal paper,[2] in which he compares properties of the ascending-bid auction and the Dutch auction (indicating possible revenue equivalence), and then introduces the second-price sealed bid auction—which has since taken his name and is, thus, often referred to as the "Vickrey auction." Many of the fundamental results in auction theory—which are often considered as part of an economist's set of basic tools—were established in the late 1970s and early 1980s, and are collected in several comprehensive surveys by McAfee and McMillan (1987), Milgrom (1989), and Wilson (1992). Since then research in auctions has dramatically expanded, with continued advances in auction theory and its applications being made by bringing the models closer to observed institutional settings. Summaries of the more recent theoretical advances are found in Wolfstetter's (1996) survey, later updated as a chapter in Wolfstetter (1999); Klemperer (1999), also reprinted in Klemperer (2004); and the more recent and very comprehensive volume by Krishna (2002).

Much of the early research on auctions went hand-in-hand with advances in game theory. However, theory was not the exclusive focus of the work done by academic economists studying auctions. Indeed, there has been independent research in the areas of empirical and experimental research which, in turn, has led to very productive cross-fertilization within the three research areas in auctions as well as in the practice of designing and conducting auctions. An excellent review of the earlier experimental work in auctions is found in Kagel (1995) which was revised and included in Kagel and Levin (2002). La□ont (1997) provides a nice survey of the earlier empirical work on auctions and Paarsch and Hong (2006) provide

[1] See, e.g., Cramton (1995), Klemperer (2004), or Milgrom (2004).

[2] Vickrey's (1961) paper now has nearly 900 citations in the Web-of-Science citation index. While there are in excess of 50 citation per year in the last six years, the article was cited fewer than once a year on average during the first fifteen years since it publication.

an up-to-date introduction and overview of structural estimation techniques for auction data. Moreover, insights gained in the study of auctions have been transferred into other areas of economic research—Klemperer (2003 and 2004) advocates this transfer of insights and gives several examples, one of which is taken up and investigated in an experimental study by Dechenaux and Mancini (2008) in the present special issue. In addition to this paper, the special issue covers a very broad range of areas, topics, and applications in modern auction research. Nevertheless, it necessarily contains only a snapshot of the larger picture, given the breadth of contemporary auction research.

The first three papers in the special issue—Harstad (2008), Lu (2008), and Mathews and Sengupta (2008)—are concerned with broadening our theoretical understanding of auction design and the implications of auction design on bidder behavior, optimal bidding, and the revenue that is generated in an auction. Mathews and Sengupta—following in the footsteps of Chwe (1989) and Rothkopf and Harstad (1994)—depart from the near-universal assumption that bids placed in an auction are freely chosen from a continuum and examine how optimal bidding strategies are a□ected in second-price, sealed-bid auctions when bids are restricted to come from a discrete grid. They show that bidders may find it optimal to place bids above their value and they establish that compared to a continuous bidding space, the choice of discrete bids available to the bidders may increase the auctioneer's revenue.

The papers by Harstad (2008) and by Lu (2008) are also concerned with the revenue generated in an auction. However, their focus is on the implications of the bidders' participation decision. They go beyond traditional models in auction theory by examining cases in which participation in the auction itself is endogenous, rather than assumed to be given. In particular, Harstad (2008) considers a framework in which bidders' entry decisions are endogenous and finds that an auctioneer who is concerned with revenue deliberately chooses entry fees that will lead in expectation to some potential bidders refraining from participating in the auction.

In a somewhat similar vein, Lu (2008) considers the revenue-maximizing auction design when bidders have private values and must incur costs in order to determine their values for the object for auction. Lu proves that in implementing the optimal auction only pure-strategy entry rules need be considered. Moreover, it is shown that the optimal scheme must entail entry fees, rather than minimum bids.

Katzman and Rhodes-Kropf (2008) is a contribution in a relatively recent and growing literature that recognizes that auctions frequently are not transactions conducted in isolation. Thus, Katzman and Rhodes-Kropf consider an auction for the right to compete in a downstream market. Since different auction formats reveal different information about bids that are placed, the auction format in place allows distinct inferences to be drawn about a bidder's type in subsequent interactions. As this affects downstream competition, bidder behavior in the auction thus becomes subject to additional strategic considerations during the auction. Interestingly, Katzman and Rhodes-Kropf also show that the announcements made in different auction formats affect bidder participation decisions, although for different reasons than those presented in Harstad (2008) and Lu (2008).

In the paper by Damianov (2008), the total amount to be sold is determined only after bidding takes place. These so-called variable-supply auctions have become more prominent in recent years as they are widely used in the auctioning of treasury bills, but are also employed in initial public offering and, recently, in wholesale electricity markets. He shows that for the case of downward sloping demand and increasing marginal costs of supply, the discriminatory auction yields the Walrasian outcome, whereas in uniform price auctions lower prices may emerge. Abrams and Schwarz (2008), while also constituting a theoretical contribution, has an immediate applied focus in considering the auctions that are conducted for advertising space on sponsored search engines on the Internet. They note that customers' experiences in clicking on ads are likely to affect their propensity to click on ads in the future. Abrams and Schwarz present an auction mechanism that leads to efficiency by internalizing these effects, when advertisers impose hidden costs on others.

Two of the papers in the special edition—Milkman, Burns, Parkes, Barron and Tumer (2008) and Dechenaux and Marcini (2008)—use laboratory experiments in their research. Milkman et al. tackle the practical issue of how auction design can facilitate bidders in converging to equilibrium behavior. In particular, they contrast a standard second-price auction format with a so-called clamped second-price auction mechanism. The latter was specifically designed in order to facilitate learning by bidders, and it outperforms the standard second-price auction in simulations with agents who are reinforcement learners. However, Milkman et al. demonstrate that bidders in a standard second-price auction actually outperform those in the clamped auction in terms of convergence to equilibrium behavior. Dechenaux and Marcini (2008), the second experimental piece in the special issue, is an

example of research into other areas of economics that draw on results and insights from auction theory. In particular, using a model on litigation rules that is based on auction theory introduced by Baye, Kovenock and de Vries (2005), they examine how subjects react when faced with di□erent litigation systems in civil suits.

The special issue concludes with two empirical papers: Bangwayo-Skeete (2008) and Brendstrup, Kuhn and Paarsch (2008). The former is a case study on the foreign exchange auctions reforms in Zimbabwe in 2004. Bangwayo-Skeete (2008) contains an overview of foreign exchange auctions, which are frequently used to allocate hard currencies in developing countries, and then discusses the particulars of the auctions used in Zimbabwe to disperse U.S. dollars. She finds that, unlike currency auctions in some comparison countries, Zimbabwe's auctions failed to stabilize currency markets largely due to poor institutional design and a lack of transparency tied to poor governance.

Brendstrup et al. (2008) is another example where research on auctions has been able to improve our understanding in areas not primarily thought of as auctions markets. Specifically, they consider a model of wage determination as a double auction and use structural estimation techniques originally developed in the auction literature in order to recover distributions of marginal productivities and opportunity costs. In applying the technique to examine data from Danish firms and workers, they find significant ine□ciencies associated with informational asymmetries between firms and workers.

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Does a Seller Really Want Another Bidder?*

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Abstract

Bulow and Klemperer [*AER*, 1996] argue that the usual concerns of auction design miss the big picture, and show that a simple English auction without a reserve price and $N + 1$ bidders attains expected revenue in excess of any auction with N bidders. The issue of how this additional bidder might be attracted is not treated in their model. In fact, that an auction can convince another bidder it is worth his while to compete carries a critical message about expected revenue. In those many markets where potential bidders decide whether to compete in an auction based on the expected profitability of bidding, Bulow and Klemperer's conclusion is shown here to be overturned. I explore the symmetric equilibrium of a model where potential bidders first decide whether to participate in an auction, and then participants select bidding strategies. Expected revenue is increased by some degree of bidder *discouragement*, in that it is never optimal to have all N potential bidders participate with probability one, even for very small N.

JEL codes: D44; D82; C72; Keywords: affiliated-values auctions, auction revenue, number of bidders, increased competition, endegenous bidder participation

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1 Introduction

Most auction theory simply posits an exogenous number of bidders without comment,[1] and proceeds to characterize such concerns as the preferences of an expected-revenue-maximizing seller across selling procedures. Bulow and Klemperer's [1996] widely noted paper indicates that the issue left out of such models may be more important than those treated:

> A seller with no bargaining power who can only run an English auction with no reserve price among $N+1$ symmetric bidders will earn more in expectation than a seller with all the bargaining power, including the ability to make binding commitments, who can hold an optimal auction with N buyers... No amount of bargaining power is as valuable to the seller as attracting one extra bona fide bidder. (p. 180)

Bulow and Klemperer are correct to broaden a seller's purview to the amount of competition an auction generates. In many auction markets, however, their prescription would be misapplied. What they compare is a basic English auction with $N+1$ bidders to an optimal auction with N bidders. In each case, the number of bidders is *exogenous*, with the bidders modeled as rational competitors, who reach the unique symmetric Bayes-Nash equilibrium.[2] For broad applicability, though, *both* the bidding once participating *and also* the (prior) decision to be a bidder ought to be presumed rationally determined.[3]

The key insight of the model presented here: an additional bidder's *deliberate* presence carries an inference about auction revenue not developed in Bulow and Klemperer's model. The additional bidder made a rational decision to enter the competition despite only 1 chance in $N+1$ of winning the auction. Hence, the profitability of being the winning bidder must be high enough to cover in expectation the resource costs of competing in $N+1$ such auctions. This higher expected profitability of competing, of course, comes largely at the expense of lower expected revenue.

The model below treats N *potential* bidders deciding whether to participate in an auction, with a symmetric equilibrium characterizing both this decision and the resultant bidding of those who do compete. An unambiguous prediction is that a seller always benefits by some measure

[1] Cf. the Klemperer [1999] survey, the Krishna [2002] text, and references cited in these works.

[2] Their assumption (A.3), p. 187, and the footnote thereto.

[3] Bidders in many markets presumably compete in or bypass an auction due to the expected profitability of bidding: oil leases, airwaves licenses, privatization of governmental enterprises, timber sales, acquisitions of distressed corporations, initial public offerings, used cars, art, and scores of others.

of *discouragement* of additional bidders. Specifically, the revenue-maximal auction never has all N potential bidders competing with probability one. This result applies even when $N = 2$ in a common-value auction, and $N = 3$ in a general affiliated-values auction. Thus, revenue may not rise with an additional bidder if his presence carries the message that he sees sufficient expected profitability to bother competing.

2 A Simple Participatory Model

Consider a model in which $N \geq 2$ is the exogenous number of *potential bidders* who will make rational decisions whether to compete in a given auction, for a single indivisible asset. The number who participate will be denoted n, and the number who actually bid denoted a.

Initially suppose the special case of a common-value auction: asset value to any potential bidder is a random variable V with density $g(v)$ bounded away from 0 and from ∞ on domain $D_V \subset [0, \infty)$. The generalization to affiliated-values auctions (for which results differ but insubstantially) is introduced after some notation.

The seller is assumed to pre-commit to an *auction mechanism* $M := (m, \varphi) \in \mathbb{M} \times \Re$, where φ is an *entry fee* (the model contemplates the possibility of a negative entry fee, which if used would be paid by the seller to a participant willing to submit a bid). Simply let $\mathbb{M}$ restrict the seller to announcing m_E, an English auction, m_2, a second-price auction, or m_1, a first-price auction. Moreover, restrict the seller to a zero reserve price.[4]

As in Bulow and Klemperer [1996], all N potential bidders are risk-neutral, and assumed to adopt symmetric strategies. A strategy for a potential bidder is $(\widetilde{\pi}_M, \kappa_n, \beta_a)_{M \in \mathbb{M}, n=1...,N, a=1...,N}$, where $\widetilde{\pi}_M$ is the probability that he participates in the auction, given that the seller has committed to auction mechanism M; κ_n is the set of his types for which he becomes an actual bidder (continuing to compete), given that n is the number of potential bidders who become *participants*; and β_a is the function of his type specifying his bidding strategy, given that a is the number of participants who became *actual bidders*.

Potential bidders simultaneously select probabilities of participating. Not participating yields a

[4] The simplifications in this and the previous paragraph are without loss of generality, as shown in results provided below or cited from Harstad [2007]. Moreover, these assumptions are conservative, in that the Bulow-Klemperer preference for an added bidder is overturned even when the seller is given a smaller set of options in the event of fewer bidders.

payoff of 0. Participation has two consequences: each participant j obtains some private information X_j about the asset's value to him (call this j's *type* or *signal*), and each incurs a *participation cost*, $c > 0$ (not revenue for the seller).[5] For simplicity, assume that the number n of participants becomes public information once simultaneous participation decisions have been made.[6]

Next, n privately informed participants make simultaneous decisions as to whether to pay the entry fee φ (or to receive it, if $\varphi < 0$). Not doing so ceases to compete, with a payoff of $-c$. Doing so makes the participant an "actual bidder."

The a actual bidders follow symmetric equilibrium bidding strategies for auction form M. The high bidder earns a payoff $V - p - \varphi - c$, where p is the price paid under rules m; other actual bidders attain payoffs $-\varphi - c$. The seller's payoff is $p + a\varphi$, price plus receipts of entry fees ("revenue" is this sum).

Each participant j draws his type X_j with continuous density $f(x|v)$ bounded away from 0 and from ∞ on domain $D_X \subset [0, \infty)$, and satisfying the maximum likelihood ratio property.[7] To avoid trivialities, $2c < E_g[V] < \infty$.[8]

To conclude this section, allow for the possibility that additional participants lead to the highest-valuing participant forecasting a higher asset value, which extends the model beyond the common-value setting. (This extension can safely be ignored by readers unconcerned about these details.) To do so, let V as used above be considered the *underlying asset value*, with asset value to a par-

[5] Hence, in this model, potential bidders who have no private information when they decide whether to become informed and compete. This makes an $N+1^{\text{st}}$ potential bidder's decision situation exactly analogous to those of the other N potential bidders. Thus, it is the natural analogue to Bulow and Klemperer's model, where the exogenously produced $N+1^{\text{st}}$ bidder is added, and his impact on expected revenue calculated, at a point where his private information is to be drawn from the same distribution as the first N bidders. In a model like Samuelson [1985], where an additional bidder incurs a resource cost to compete after freely acquiring his private information, the impact on expected revenue would depend entirely on the relation between that additional bidder's type and the two highest of the original bidders' types. Similarly, Bergemann and Välimäki [2002] consider the issue of additional information acquisition by bidders who already possess imperfect but private information about their valuations. In their model, the number of bidder is exogenous; the impact of an additional bidder would depend on the likelihood that he possessed one of the two highest valuations.

[6] Harstad [2007] also treats the case where the number of participants is not learned, with identical results. The issue of simultaneity of the participation decision is considered in concluding remarks.

[7] The common and seemingly innocuous assumption that participants observe asset value estimates of *symmetric precision*—which this model shares with Bulow and Klemperer—actually has a subtle but important impact: it means that an additional participant and additional information about asset value enter simultaneously, and together alter revenue. This impact is first considered in Mares and Shor [2007], who point out that Bulow and Klemperer do not separate the revenue impact of added competition from the revenue impact of added information. They build a model handling asymmetrically precise value estimates, so that an (exogenous) additional bidder can be considered in a context where the added competition occurs with the aggregate set of signals unaltered, and provide assumptions under which an exogenously added bidder without any addition to the aggregate information still leads to increased expected revenue. In the current model, an additional participant arrives with additional information, and the combination may lead to lower revenue. I point in the concluding discussion to the difficulty of assessing endogenous arrival of additional competition without assuming symmetric information precision.

[8] These assumptions are drawn from Pesendorfer and Swinkels [1997]; cf. the lengthier discussion there.

ticular participant observing signal X_i a continuous function $t(V, X_i)$, increasing in both variables (common across participants, in that t does not have a subscript). Without loss of generality, presume $t(V, X_i) \geq V \ \forall X_i \in \mathcal{X}$; the common-value setting will be the special case $t(V, X_i) = V$. The winning bidder's payoff becomes $t(V, X_i) - p - \varphi - c$; other payoffs are unchanged.

3 Equilibrium Revenue

In symmetric equilibrium for auction mechanism M, all potential bidders select the same probability of participating, $\pi(M) \geq 0$. Hence, $\pi(M) \in (0, 1)$ must leave any given potential bidder indifferent over participating when all $N-1$ other potential bidders participate with probability $\pi(M)$. This requirement may informally be expressed as $w(\pi) E[T(M, \pi, n, \varphi, v) - p(M, \pi, a, n)] - \varphi e(M, \pi, a, n) - c \geq 0$, with strict inequality permitted only if $\pi = 1$. In this equation, $w(\pi)$ is the ex-ante probability that a potential bidder wins conditional on his participating; $T(M, \pi, n, \varphi, v)$ is the expected asset value to a potential bidder who will participate, conditional on his winning, and on π participation probability, n participants, φ entry fee, and a draw of v from $g(\cdot)$; $p(M, \pi, a, n)$ is an equilibrium expected price paid by the winning bidder; and $e(\cdot)$ is the ex ante probability that a potential bidder pays the entry fee conditional on his participating. This latter term, $e(M, \pi, a, n)$, sums (with Bernoulli weights for the probabilities of $n-1$ rival participants) over the probability of drawing a type for which φ is not greater than the product of expected profitability given winning with type x times the probability of winning with type x against $n-1$ rival participants.[9]

With a zero reserve price, the auction will see a transaction every time at least one participant is willing to pay the entry fee. With the seller's value of the asset normalized to zero, which is the minimum of the support of asset value to a participant, every transaction yields a gain from trade. The expected value of the gains from trade for mechanism M is thus

$$\overline{V}(M) = \sum_{1 \leq n \leq N} \sum_{1 \leq a \leq n} E[T[M, \pi(M), n, \varphi, V] \mid n, \{a > 0\}] \binom{n}{a} \mu_a(M, n) \beta_n,$$

where expected asset value is conditioned on at least one of n participants electing to pay the entry fee; $\mu_a(M, n)$ is the ex ante probability, conditional on n participants, that participants $1, \ldots, a$ choose to pay the entry fee, and $a+1, \ldots, n$ choose not to; and β_n is the probability of n participants

[9] Details are laid out for a more general model in Harstad [2007], section 3. Note that $T(\cdot)$, $p(\cdot)$ and $e(\cdot)$ will typically be degenerate in at least one variable for a given m.

given the equilibrium value of π.

The expected number of actual bidders is

$$\overline{a}(M) = \sum_n \left\{ \sum_a a \binom{n}{a} \mu_a(M,n) \right\} \beta_n$$

and the expected number of participants is $\overline{n}(M) = \sum_n n\beta_n = N\pi(M)$.

Expected revenue sums the price and expected entry fee receipts: $\overline{R}(M) = p(M,\pi,a,n) + \overline{a}(M)\varphi$.

If each participant is indifferent between participating or not, then equilibrium requires the expected gains from trade to accrue to the seller, net of effectively compensating participants for their resource costs. This conclusion can be expressed as

$$\overline{R}(M) = \overline{V}(M) - c\overline{n}(M), \tag{1}$$

and has several antecedents in the auction literature.[10]

The seller can attain the entire interval of equilibrium participation probabilities, 0 through 1. Setting $\varphi = E[V]$ yields $\pi(M) = 0$, and setting $\varphi = -c$ yields $\pi(M) = 1$. Intermediate values are obtained from having enough continuity of expected profitability in the entry fee to apply the intermediate value theorem.[11]

A key characteristic of the symmetric equilibrium yielding (1) is that two different auction mechanisms attaining the same equilibrium participation probability π attain the same level of equilibrium revenue. Indeed, there exists a function $R(\pi)$ on $[0,1]$ with the property that $\overline{R}(M) = R(\pi(M))$.[12]

4 Discouraging Competition

The desired characterization is simply stated and shown. That $R(\pi)$ is continuous is straightforward. As its range is obviously bounded, it attains a maximum. Let π^H be the largest π attaining

[10] Theorem 3 in Harstad [2007] encompasses as special cases corresponding prior results in Samuelson [1985], Hausch and Li [1990], Harstad [1990] and Levin and Smith [1994], with presursors in Milgrom [1981], Matthews [1984], and French and McCormick [1984]. See also footnotes 19 and 20.

[11] A proof for the case of the second-price auction is Thm. 5 in Harstad [2007].

[12] This characteristic is a corollary of Proposition 7, and the projection onto $[0,1]$ to yeild $R(\pi)$ is Corollary 8, in Harstad [2007].

this maximum.

Proposition 1 $\pi^H < 1$ *in a common-value auction, or more generally if* $N \geq 3$.

Proof. Consider $R(\pi)$ as π decreases incrementally from 1 to $1 - \triangle\pi$. This has three analytically separable effects on expected revenue. First, seller in equilibrium compensates bidders for their participation costs, and so this effect yields a gain in expected revenue as aggregate expected participation costs decrease. Second, the highest (across participants) expected value of the auctioned asset falls, as this no longer attaches probability one to the highest of N values (with small probability it becomes the highest of $N-1$ values). To measure this effect, define $J(n) = E_\xi[\max_{i=1,...,n} t(V, X_i)]$; then this expected highest value is

$$K(\pi) = \sum_{n=1,...,N} \left(\begin{array}{c} N \\ n \end{array} \right) \pi^n (1-\pi)^{N-n} J(n).$$

Third, the probability of 0 actual bidders increases, reducing expected revenue as there are then no gains from trade for seller to appropriate. As π decreases very near 1, the first effect shows a revenue gain that is linear in $\triangle\pi$, by (1); the second effect is absent under pure common values, and otherwise a revenue loss that is of the order $(\triangle\pi)^{N-1}$, by derivation of $\partial K/\partial\pi|_{\pi=1-\triangle\pi;}$ and the third effect a revenue loss that is of the order $(\triangle\pi)^N$. For small $\triangle\pi$, the first effect dominates. ■

Hence, a seller prefers to discourage potential bidders from competing at least to some extent: relative to any revenue-maximal mechanism, there always exists an alternative auction mechanism exhibiting a larger expected number of participants. That is, a seller wishes to make the auction mechanism sufficiently extractive of surplus that it is not in a bidder's interest to participate if all rival potential bidders are participating with probability 1. Notice that in a common-value auction with $N = 2$ potential bidders, rather than an auction yielding $\pi = 1$, the seller will have some revenue-superior alternative which will lead potential bidder 1 to be indifferent over participating even when he infers that there will be at least a $\left(1 - \pi^H\right) > 0$ probability of facing no competition (and a zero reserve price). With a private-values element (a general affiliated-values setting), a seller never prefers to attract a third participant with probability one.[13]

[13] An illustrative example with $N = 2$: let $t(V, X_i) = X_i$ with $X_i \sim U[0,1]$ (uniformly distributed independent private values). Then $sign(\partial R(\pi)/\partial\pi|_{\pi=1-\triangle\pi}) = sign((1/6) - c)$, from (1). As at $\pi = 1$ and $\varphi = 0$, a participant's

This discouragement of potential bidders might possibly be handled by switching from a first-price to a second-price auction, or from either to an English auction. Each of these switches in auction form, and any other change that would extract more surplus from a number of bidders fixed exogenously, serves in this model to reduce π.[14] While each of the switches in auction form has a discrete impact on π, and might reduce it too far, π responds continuously to changes in the entry fee.[15]

5 Concluding Discussion

Several auction papers have found expected revenue for a given auction form to be increasing in the number of competitors.[16] What is striking about the Bulow and Klemperer result is that this effect of an exogenously specified arrival of further competition dominates all aspects of auction design, making the simple English auction with an additional bidder superior to any auction with a given level of competition.

However, this paper has offered a model for situations where the arrival of additional competition is itself a result of rational decision, and the seller rationally takes into account the inference that the additional participant is there *deliberately*.[17] This inference, that the asset is expected to sell for far enough below its value as to make a smaller chance of winning ($1/[n+1]$ instead of $1/n$) still worthwhile, overturns the prior preference for more competition.[18] Some degree of discouragement

equilibrium expected profitability is 1/6, a seller will only attain a higher expected payoff at $\pi = 1$ than at $\pi = 1 - \triangle\pi$ if this payoff includes positive entry fee receipts, and thus will in this sense still be discouraging bidder participation.

[14] This result is an interesting complement to Bergemann and Pesendorfer [2007]. They consider an exogenously given number of bidders, and allow a seller to control both the allocation mechanism and the precision with which bidders learn their (independent) private values. Here, the precision of competitors' private information is exogenous, but seller through the allocation mechanism affects their incentives to acquire information.

[15] The proof in Theorem 5 in Harstad [2007] for the second-price auction is readily imitated for an English auction. Maskin and Riley [2000] establish a corresponding degree of continuity for a first-price auction.

[16] This is a direct implication in the theoretical models of Friedman [1956], Rothkopf [1969], Myerson [1981], Harris and Raviv [1981] and Riley and Samuelson [1981], and an asymptotic result in Wilson [1977], Pesendorfer and Swinkels [1997] and Harstad, Tsetlin and Pekeč [2008]. Simulations by Reece [1978] and empirical studies by Mead [1969], Smith [1977], Gaver and Zimmerman [1977], Hendricks, Porter and Boudreau [1987], and Giliberto and Varaiya [1989], among others (see surveys by Laffont [1997] and Porter [1995]), show this comparative static.

[17] Under the title, "What Really Matters in Auction Design," Klemperer [2002], after discussing collusion, states "The second major area of concern of practical auction design is to attract bidders, since an auction with too few bidders risks being unprofitable for the auctioneer." (p. 172) However, in discussing a variety of examples, espeically airwaves auctions, Klemperer frequently recognizes the need for an additional bidder to see sufficient chance of winning in order to be attracted to compete, and mentions in his conclusions "even modest bidding costs may be a serious deterrent to potential bidders." (p. 186) The model presented here helps to formalize assertions implicit in his discussion of some of these examples.

[18] A similar inference can be drawn in the much more involved model of Crémer, Spiegel and Zheng [2006]. They allow a seller to limit competition by having the capability to control when particular competitors learn their private

of potential bidders is beneficial: Proposition 1 shows the seller always has an option with a *higher* expected number of competitors than the expected-revenue-maximizing option.[19]

It is clear that this inference to be drawn from added competition that deliberately competes is more general than the simple model used here to illustrate it. One particular case is where added bidders arise via a sequential decision process as to whether to participate. The presence of the second participant implies that the expected profitability of being the winning bidder is at least twice the participation cost, when the price is determined by competition among two. The presence of a third participant would imply that the expected profitability of being the winning bidder is at least three times the participation cost, when the price is determined by competition among three. That higher level of expected profitability, of course, comes primarily at the seller's expense.

The assumption of symmetry in potential bidders' information, beliefs and behavior is critical to the above analysis, as it generates a unique equilibrium continuation for any auction mechanism. This uniqueness makes expected profitability calculations well-defined; without it, expected profitability calculations governing whether to participate and whether to pay the entry fee would depend on which of many (typically infinitely many) equilibrium continuations might follow.

Nonetheless, in many situations characterized by asymmetric sequential decisions about whether to compete in an auction or other contest, it may be reasonable that participants arrive in decreasing order of expected profitability of participating. That would suggest the negative inference

values. It is then often optimal to permit a single buyer to incur the cost of learning his private value, approach him with a take-it-or-leave it price, and continue on to allowing one more buyer to learn his private value if that price is rejected. In this sense, added competition is essentially a last resort in their optimum. Their intriguing result clearly depends on a stringent assumption that the seller can in effect impose a bidder-specific lump-sum tax on all (potential) bidders at the beginning of the game. The current model shares with most auction theory papers the tradition of assuming that any surplus-extracting device a seller can employ is potentially distortive of bidders' behavior.

[19] The contrast with Bulow and Klemperer is relatively straightforward. More complex is the relationship of this result with the interesting characterizations in McAfee [1993]. McAfee builds a partly game-theoretic, partly competitive-equilibrium model of an outcome roughly akin to a steady state in an infinitely recurring market. Each seller auctions a single asset with a reserve price in each period; each independent-private-values buyer selects a seller's auction to compete in based on the profile of reserve prices. (In the steady state, all sellers select the same reserve price, and all buyers use the uniform mixed strategy to select an auction to attend.) A transaction leads to the seller and the successful buyer exiting the market. Each period then sees a replenishment of new sellers and new buyers in an exogenous ratio. McAfee's characterizations of the steady-state in-period reserve-price choice as efficient, and of the resulting steady-state expected revenue, are fully consistent with equation (1) above. An exogenous increase in the ratio of newly arriving buyers per arriving seller is beneficial to sellers in the steady state. This is partly because these added arriving buyers are exogenously specified; an arriving buyer's attendance at a particular seller's auction is endogenous, but his arrival at the market of competing sellers is not the result of any decision by him. It is also partly due to each seller responding to the higher buyer/seller ratio by increasing his in-period reserve-price, but this happens without any impact on the number of buyers arriving at his auction, as all other sellers have identically increased their reserve price in the altered steady-state associated with the exogenous change in the buyer/seller ratio. The comparative static in this model that would correspond to all rival auctions becoming more extractive of surplus would be a reduction in c. The opportunity cost of competing in this auction, which will be lower if rival auctions have, for example, higher entry fees, is reflected in c. Clearly, a lower c benefits the seller.

associated with additional competitors developed here may be all the more dramatic.[20]

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Optimal Entry in Auctions with Valuation Discovery Costs[†]

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Abstract This paper further studies auction design in an independent private value (IPV) setting where bidders have fixed valuation discovery costs. Our contributions are two-fold. First, we develop a convenient way of addressing this auction design problem with endogenous entry, which accommodates all entry possibilities. Second, we find that there is no loss of generality in considering entry patterns where every bidder participates with probability of either 0 or 1, for the revenue-maximizing (meanwhile ex ante efficient) auction. As entrants generally enjoy positive surplus at the optimal discrete entry, the revenue-maximizing auction generally needs to employ positive entry fees to extract the surplus of them.

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JEL classifications: D44, D82.

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1 Introduction

The impact of valuation discovery costs on bidders' entry decisions and auction design, has been extensively studied in the literature.[1] Here, valuation discovery costs refer to the costs that bidders must incur to discover their valuations of the auctioned object.[2] In this paper, we further study auction design and optimal entry in an environment where the value discovery costs are fixed.[3] In particular, we study the optimal entry in terms of seller's expected revenue and expected total surplus of seller and bidders. The feature of our analysis lies in that we allow bidders to participate with any probabilities in $[0, 1]$. This feature differentiates our study from the existing literature. McAfee and McMillan (1987) and Engelbrecht-Wiggans (1993) have focused on discrete entry of bidders, while Levin and Smith (1994) and Ye (2004) look at the symmetric mixed strategy entry equilibrium in symmetric settings.

As we allow bidders to participate with any probabilities in $[0, 1]$, these entry probabilities can be asymmetric across bidders. Our analysis with this additional flexibility results in new findings. **First**, there is no loss of generality in considering the entry patterns where every bidder participates with probability of either 0 or 1 for the revenue-maximizing (meanwhile ex ante efficient) auction. This result eliminates the necessity of considering mixed strategies of bidders for analysis on optimal entry. The implication of this result is two-fold: (i) This finding greatly simplifies the auction design problems; (ii) the actual number of participants is always discrete in reality. Our result means that this causes no loss to optimal auction design. **Second**, our result means that the revenue-maximizing entry is generally an asymmetric corner solution rather than a symmetric inner solution in terms of entry probabilities. This explains why seller's expected revenue may drop with the number of potential bidders when entry is restricted to be symmetric in Levin and Smith (1994). **Third**, a second-price auction with no entry fee and a reserve

[1] Bergemann and Välimäki (2005) present a thorough review of the literature.

[2] The valuation discovery cost should be distinguished from the entry cost that the bidders who know their valuations must incur in order to prepare for their bids.

[3] This setting of fixed costs is adopted by McAfee and McMillan (1987), Engelbrecht-Wiggans (1987, 1993), Harstad (1990), Levin and Smith (1994) and Ye (2004) among others.

price equal to seller's value remains to be ex ante efficient though the optimal entry can be asymmetric across bidders. **Fourth**, as entrants generally enjoy positive payoffs at discrete entry, entry fees generally are essential to extract all the expected surplus of the participants for revenue maximization.

2 The Model

We will follow Levin and Smith (1994) and work with the following symmetric IPV setting.[4] There are N potential bidders who are interested in a single item auctioned, where N is public information. Denote this group of potential bidders by $\mathcal{N} = \{1, 2, \cdots, N\}$. The seller's value is v_0, which is public information. Every bidder has to incur a fixed entry cost of $c(> 0)$ in order to enter the auction. After incurring an entry cost c, he observes his private value v_i, where v_i is the private information of bidder i.[5] The cumulative distribution function of $v_i,\ \forall i \in \mathcal{N}$ is $F(\cdot)$ with density function of $f(\cdot)$ defined on $[\underline{v}, \overline{v}]$. The distribution of $v_i,\ i \in \mathcal{N}$ are public information. We assume $v_i,\ \forall i \in \mathcal{N}$ are mutually independent. The seller and bidders are risk neutral.

The game unfolds in the following sequence: (i) All public information is revealed by Nature; (ii) The seller announces the rule of the auction; (iii) The bidders simultaneously and confidentially make their entry decisions. If they do not enter, they take the outside option which gives them zero payoffs. If they enter, they incur their entry costs and observe their private values. (iii) Every participant bids;[6] (iv) The payoffs of the seller and all the participating bidders are determined according to the announced rule.

We study the auction rules, which maximize the expected total surplus and the seller's

[4]it is straightforward to extend our analysis to asymmetric settings.

[5]The assumption that bidders have to incur the cost to enter is well adopted in the literature. However, this assumption precludes the possibility that a bidder may simply submit a bid equal to the unconditional expected value. When $Ev_i < v_0$, the assumption is innocent as the buyers who do not know their values prefer not to bid in the revenue-maximizing auction that will be presented in Proposition 3.

[6]Every participant may or may not observe the other participants. The auctions designed later work in both cases.

expected revenue. Hereafter, the auction maximizing expected total surplus of seller and bidders is called the **ex ante efficient auction**; the auction maximizing seller's expected revenue is called the **revenue-maximizing auction**.

3 Auction Design

Our analysis allows every possible entry pattern which can be described through a vector of entry probabilities $\mathbf{P} = (p_1, p_2, \cdots, p_N)$ where $p_i \in [0,1], \forall i \in \mathcal{N}$ is bidder i's entry probability. For convenience, we use $\mathcal{A}_\mathbf{0}$ to denote the second-price auction with no entry fee and a reserve price equal to the seller's value v_0. We first establish the following results.

Lemma 1: *Among all auctions implementing any given entry pattern* $\mathbf{P} = (p_1, p_2, \cdots, p_N)$*, a second-price auction with a reserve price equal to seller's value and appropriately set ex ante entry fee (or subsidy) for each bidder provides the highest seller's expected revenue as well as the highest expected total surplus. The ex ante entry fees (or subsidies) are charged before the entrants learn their values, and are set at levels such that each entrant gets zero expected payoff at the equilibrium.*[7]

Proof: Let us first consider auction $\mathcal{A}_\mathbf{0}$. Suppose all bidders other than i participate in auction $\mathcal{A}_\mathbf{0}$ according to their entry probabilities $\mathbf{P}_{-i} = (p_1, \cdots, p_{i-1}, p_{i+1}, \cdots, p_N)$. Denote bidder i's expected surplus by $\pi_i(\mathbf{P}_{-i})$ if he participates in $\mathcal{A}_\mathbf{0}$. Set an ex ante entry fee (or subsidy) for bidder i as $e_i = \pi_i(\mathbf{P}_{-i}),\ \forall i \in \mathcal{N}$. Clearly, for a second-price auction with entry fee (or subsidy) e_i for bidder i and a reserve price equal to seller's value, bidder i is indifferent between participation and not. Therefore, the above auction with entry fee e_i for bidder i, $\forall i \in \mathcal{N}$ implements entry $\mathbf{P}$. Every bidder has an expected payoff of zero. Note that for any auction implementing $\mathbf{P}$, the total expected entry costs are the same. Thus this auction achieves the highest possible expected total surplus among all auctions implementing $\mathbf{P}$, since the auction always allocates the item to the participant

[7] If $p_i = 0$, the entry fees for bidder i can be set any level such that he gets nonpositive payoff when participating.

(including the seller) with the highest value. Moreover, for any auction implementing entry pattern $\mathbf{P}$, the expected payoff of bidder i cannot be smaller than 0 due to the participation constraint. As a result, the proposed auction achieves the highest possible seller's expected revenue among all auctions implementing entry $\mathbf{P}$. □

In Lemma 1, the seller is allowed to charge ex ante entry fees before buyers check the object. This has been pointed out by McAfee and McMillan (1987): "... the seller must extract these payments ... before any bidder learns his valuation: this could be done, for example, by demanding a fee before potential bidders inspect the item for sale." Engelbrecht-Wiggans (1993) and Levin and Smith (1994) also include ex ante entry fees in their analysis. We will show later that ex ante entry fee turns out to be redundant for the ex ante efficient auction we will establish. For revenue maximization, the proposed mechanism would rather require ex ante entry fee.

According to Lemma 1, for a given entry pattern $\mathbf{P}$, the auction that maximizes seller's expected revenue also maximizes expected total surplus.[8] Since the expected payoffs of bidders are zero at the optimum, the seller's optimal expected revenue equals the optimal expected total surplus. Denote this optimal expected total surplus by $S(\mathbf{P})$. We next introduce a way of writing $S(\mathbf{P})$ that will greatly facilitate our analysis and the interpretation of the results.

We define set $\mathcal{K} = \{(k_1, k_2, \cdots, k_N) | k_i \in \{0, 1\},\ i \in \mathcal{N}\}$, where k_i denotes bidder i's ex post entry status. Specifically, $k_i = 1$ stands for the participation of bidder i, while $k_i = 0$ represents the non-participation of bidder i. In addition, $k_0 \equiv 1$ symbolizes the participation of the seller. For any $\mathbf{k} = (k_1, k_2, \cdots, k_N) \in \mathcal{K}$, we use $v_{\mathbf{k}}$ to denote the highest value of all ex post participants including the seller, i.e., $v_{\mathbf{k}} = \max_{\{k_j=1, 0 \le j \le N\}}\{v_j\}$. We use $f_{\mathbf{k}}(v_{\mathbf{k}})$ and $F_{\mathbf{k}}(v_{\mathbf{k}})$ to denote the density and cumulative distribution functions of $v_{\mathbf{k}}$, respectively. Furthermore, $V_{\mathbf{k}}$ denotes the expectation of $v_{\mathbf{k}}$. $S(\mathbf{P})$ can then be written as

$$S(\mathbf{P}) \quad = \quad \sum_{\{\mathbf{k} \in \mathcal{K}\}} \left(V_{\mathbf{k}} \prod_{i \in \mathcal{N}} p_i^{k_i} (1 - p_i)^{1 - k_i} \right) - \sum_{i \in \mathcal{N}} p_i c. \tag{1}$$

The first summation term in (1) is the contribution of the values of all participants

[8] Note that Proposition 1 in Levin and Smith (1994) has shown this result for symmetric entry.

including the seller to the expected total surplus if potential bidders participate according to probabilities $\mathbf{P}$. The second term corresponds to the contribution (negative) of their entry costs. Clearly, $S(\cdot)$ is a continuous function defined on a compact set $\mathcal{D} = [0,1]^N$. Thus, there must exist $\mathbf{P}^* = (p_i^*) \in \mathcal{D}$ that maximizes $S(\cdot)$ within $\mathcal{D}$.

Define $\mathcal{K}_{-i} = \{(k_1, \cdots, k_{i-1}, k_{i+1}, \cdots, k_N) | k_j \in \{0,1\},\ j \neq i.\},\ \forall i \in \mathcal{N}$. Let us consider bidder i's optimal entry probability $p_i^*,\ \forall i \in \mathcal{N}$. For any $\mathbf{k}_{-i} = (k_1, \cdots, k_{i-1}, k_{i+1}, \cdots, k_N) \in \mathcal{K}_{-i}$, we use $\mathbf{k}_1(\mathbf{k}_{-i})$ to denote the N-element vector where the i-th element is 1 and other elements are same with $\mathbf{k}_{-i}$, while we use $\mathbf{k}_0(\mathbf{k}_{-i})$ to denote the N-element vector where the i-th element is 0 and other elements are same with $\mathbf{k}_{-i}$. We then have

$$\frac{\partial S(\mathbf{P})}{\partial p_i} = \sum_{\{\mathbf{k}_{-i} \in \mathcal{K}_{-i}\}} \{(V_{\mathbf{k}_1(\mathbf{k}_{-i})} - V_{\mathbf{k}_0(\mathbf{k}_{-i})} - c) \prod_{j \neq i} p_j^{k_j} (1-p_j)^{1-k_j}\}, \tag{2}$$

as $\sum_{\{\mathbf{k}_{-i} \in \mathcal{K}_{-i}\}} (\prod_{j \neq i} p_j^{k_j} (1-p_j)^{1-k_j}) = 1$. Note that $\frac{\partial S(\mathbf{P})}{\partial p_i}$ does not depend on p_i. Thus, $S(\mathbf{P})$ is a linear function of p_i given $\mathbf{P}_{-i}$, and the slope only depends on $\mathbf{P}_{-i}$.

Suppose $\mathbf{P}^*$ is an entry pattern that maximizes $S(\mathbf{P})$, we must have for all $i \in \mathcal{N}$,

$$\frac{\partial S(\mathbf{P}^*)}{\partial p_i} = \begin{cases} 0, & \text{if} \quad p_i^* \in (0,1), \\ \geq 0, & \text{if} \quad p_i^* = 1, \\ \leq 0, & \text{if} \quad p_i^* = 0. \end{cases} \tag{3}$$

Define $S_i(\mathbf{P}_{-i}) = \sum_{\{\mathbf{k}_{-i} \in \mathcal{K}_{-i}\}} \{(V_{\mathbf{k}_1(\mathbf{k}_{-i})} - V_{\mathbf{k}_0(\mathbf{k}_{-i})} - c) \prod_{j \neq i} p_j^{k_j} (1-p_j)^{1-k_j}\}$, which is the right hand side of (2). Note that the economic meaning of $V_{\mathbf{k}_1(\mathbf{k}_{-i})} - V_{\mathbf{k}_0(\mathbf{k}_{-i})} - c$ is the marginal contribution of bidder i to the expected total surplus if he participates in auction $\mathcal{A}_{\mathbf{0}}$, and all other participants are those bidders with $k_j = 1,\ j \neq i$ in vector $\mathbf{k}_{-i}$. Hence, $V_{\mathbf{k}_1(\mathbf{k}_{-i})} - V_{\mathbf{k}_0(\mathbf{k}_{-i})} - c$ can be alternatively written as

$$\int_{v_0}^{\overline{v}} \Big\{(v_i - v_0) F_{\mathbf{k}_0(\mathbf{k}_{-i})}(v_0) + \int_{v_0}^{v_i} (v_i - v) f_{\mathbf{k}_0(\mathbf{k}_{-i})}(v) dv\Big\} f(v_i) dv_i - c, \tag{4}$$

where $F_{\mathbf{k}_0(\mathbf{k}_{-i})}(\cdot)$ and $f_{\mathbf{k}_0(\mathbf{k}_{-i})}(\cdot)$ are the cumulative distribution function and density function of $v_{\mathbf{k}_0(\mathbf{k}_{-i})}$, respectively. (4) can also be interpreted as the expected payoff of bidder i when he participates in auction $\mathcal{A}_{\mathbf{0}}$, if all other participants are those bidders with $k_j = 1,\ j \neq i$ in vector $\mathbf{k}_{-i}$.[9] It then follows that $S_i(\mathbf{P}_{-i})$ is the expected payoff of bidder

[9]It is a weakly dominant strategy for entrants to bid their true values in a second price auction.

i when he participates in auction $\mathcal{A}_\mathbf{0}$, if all other potential bidders participate according to $\mathbf{P}_{-i}$. This insight together with (2) and (3) lead to the following proposition.

Proposition 1: *Suppose* $\mathbf{P}^*$ *is an entry probability vector that maximizes* $S(\mathbf{P})$ *within* $\mathcal{D}$. *(i) Auction* $\mathcal{A}_\mathbf{0}$ *is ex ante efficient. (ii) A second-price auction with a reserve price equal to seller's value* v_0 *and an ex ante entry fee* e_i^* *for bidder* i *maximizes the seller's expected revenue, where* $e_i^* = S_i(\mathbf{P^*}_{-i}) = 0$ *if* $p_i^* \in (0,1)$*;* $e_i^* = S_i(\mathbf{P^*}_{-i})(\geq 0)$ *if* $p_i^* = 1$*; and* e_i^* *can be any number no less than* $S_i(\mathbf{P^*}_{-i})(\leq 0)$ *if* $p_i^* = 0$.

Proof: Clearly, auction $\mathcal{A}_\mathbf{0}$ implements entry $\mathbf{P}^*$. For second price auction $\mathcal{A}_\mathbf{0}$, every participant bids his value, thus the participant (including the seller) with the highest value wins. As a result, $\mathcal{A}_\mathbf{0}$ renders the highest possible expected total surplus $S(\mathbf{P}^*)$.

The second price auction defined in Proposition 1 (ii) also implements entry $\mathbf{P}^*$ and achieves the expected total surplus $S(\mathbf{P}^*)$. Since it extracts all the surplus of the participating bidders, it is revenue-maximizing for the seller. □

We now turn to the properties of the optimal entry $\mathbf{P}^*$. The following proposition shows that there must exist a discrete entry that is optimal.

Proposition 2: *There must exist an entry probability vector* $\mathbf{P}^*$ *where* $p_i^* \in \{0,1\}, \forall i \in \mathcal{N}$*, which maximizes* $S(\mathbf{P})$. *In other words, there is no loss of generality to consider only pure-strategy entry for efficient and/or revenue-maximizing auctions.*

Proof: Suppose $\mathbf{P^{*(0)}} = (p_1^{*(0)}, \cdots, p_N^{*(0)})$ maximizes $S(\mathbf{P})$, where $p_i^{*(0)} \in [0,1], \forall i \in \mathcal{N}$. If $p_i^{*(0)} \in \{0,1\}, \forall i \in \mathcal{N}$, then the proof is completed. Otherwise, there must exist $i_0 \in \mathcal{N}$ such that $p_{i_0}^{*(0)} \in (0,1)$. Since $p_{i_0}^{*(0)}$ is an inner solution, we must have $\frac{\partial S(\mathbf{P^{*(0)}})}{\partial p_{i_0}} = S_{i_0}(\mathbf{P^{*(0)}}_{-i_0}) = 0$ from (3). As pointed out before, $S(\mathbf{P})$ is a linear function of p_i for given $\mathbf{P}_{-i}$, and the slope is $S_i(\mathbf{P}_{-i})$ that only depends on $\mathbf{P}_{-i}$. Therefore, when $S_{i_0}(\mathbf{P^{*(0)}}_{-i_0}) = 0$, $S(\mathbf{P^{*(0)}})$ actually only depend on $\mathbf{P^{*(0)}}_{-i_0}$ but not $p_{i_0}^{*(0)}$. Thus, setting $p_{i_0}^{*(0)}$ to be either 0 or 1 will not change $S(\cdot)$. Thus, the resulting entry $\mathbf{P^{*(1)}}$ from this change in $p_{i_0}^{*(0)}$ must also maximize $S(\mathbf{P})$, where $\mathbf{P^{*(1)}}$ differs from $\mathbf{P^{*(0)}}$ only in its i_0th element. It follows that (3) must also hold for $\mathbf{P^{*(1)}}$, as it also maximizes $S(\cdot)$. If now $p_i^{*(1)} \in \{0,1\}, \forall i \in \mathcal{N}$, then the proof is completed. Otherwise, the above process can continue until every element in the entry probability vector becomes either 0 or 1. □

Define $\pi(n) = S_i(\mathbf{P}_{-i})$ where $\mathbf{P}_{1\times n} = (1,1,\cdots,1)$. $\pi(n)$ is then the marginal con-

tribution of a participant to the expected total surplus in auction $\mathcal{A}_\mathbf{0}$ if there are other $n-1$ participating bidders. Clearly, $\pi(n)$ decreases with n and turns negative for big n. If $\pi(1) < 0$, define $n^* = 0$, otherwise define $n^* = \max_{\{\pi(n)\geq 0\}}\{n\}$. Based on Proposition 2, integer n^* (≥ 0) must satisfy the following property. If $N \leq n^*$, it is optimal for all bidders to participate with probability 1. If $N \geq n^*$, it is optimal that any n^* bidders participate with probability of 1, and other bidders participate with probability zero.

Levin and Smith (1994) show that expected total surplus may decrease with the number of potential bidders when the entry patterns are restricted to be symmetric across bidders. Proposition 2 implies that if no restriction is imposed on the entry pattern, n^* is the optimal number of bidders if $N > n^*$. Thus, additional potential bidders beyond n^* neither increases nor decreases the seller's optimal expected revenue. Consider an example where $v_0 = 0,\ c = 0.1,\ F(v) = v^2$ on $[0, 1]$. Calculation shows that when $N = 2$, $\mathbf{P}^* = (1, 1)$ and $S(\mathbf{P}^*) = 0.60$. Note that the entry is symmetric. When $N = 3$, $\mathbf{P}^* = (1, 1, 0)$ and $S(\mathbf{P}^*) = 0.60$. In addition, $\mathbf{P}^\dagger = (.83, .83, .83)$ is the optimal symmetric entry, which gives $S(\mathbf{P}^\dagger) = 0.57017$. This example shows that symmetric entry in mixed strategy generally cannot be optimal.

Suppose $N > n^* \geq 1$ and $\tilde{\mathbf{P}}^*$ maximizes $S(\cdot)$, where every element $\tilde{p}_i^*$ of $\tilde{\mathbf{P}}^*$ is either 0 or 1. There must be n^* elements in $\tilde{\mathbf{P}}^*$, which are 1. Without loss of generality, we can assume the first n^* elements in $\tilde{\mathbf{P}}^*$ are 1. From Proposition 1, we have the following result. The proof is omitted.

Proposition 3: *A second-price auction with a reserve price equal to seller's value and a uniform ex ante entry fee $e^* = S_1(\tilde{\mathbf{P}}^*_{-1})$ is revenue-maximizing.*

The auction of Proposition 3 also induces an inefficient symmetric entry in mixed strategy. Thus, coordination devices such as sequential entry decisions are usually necessary for the auction to implement the pure-strategy efficient entry.

The seller can also uniquely induce the revenue-maximizing pure-strategy entry through limiting the number of potential bidders to n^*. For example, he can offer a second price auction with a reserve price equal to the seller's valuation and a uniform entry fee $e^* = S_1(\tilde{\mathbf{P}}^*_{-1})$ to any group of n^* potential bidders.

4 Concluding Remarks

This paper develops a convenient way of addressing auction design with valuation discovery costs. Our analysis accommodates all possibilities of endogenous entry of bidders. We find that there is no loss of generality in considering discrete pure-strategy entry for the revenue-maximizing (or ex ante efficient) auction. As entrants generally enjoy positive payoffs at discrete entry equilibrium, individual entry fees are necessary to extract the bidders' surplus for revenue maximization. The revenue-maximizing auction thus takes a form of a second price auction with a reserve price equal to seller's valuation and an (ex ante) entry fee.

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Sealed Bid Second Price Auctions with Discrete Bidding*

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Abstract

A single item is sold to two bidders by way of a sealed bid second price auction in which bids are restricted to a set of discrete values. Restricting attention to symmetric pure strategy behavior on the part of bidders, a unique equilibrium exists. When following these equilibrium strategies bidders may bid strictly above or below their valuation, implying that the item may be awarded to a bidder other than the high valuation bidder. In an auction with two acceptable bids, the expected revenue of the seller may be maximized by a high bid level not equal to the highest possible bidder valuation and may exceed the expected revenue from an analogous second price auction with continuous bidding (and no reserve price). With three acceptable bids, a revenue maximizing seller may choose unevenly spaced bids. With an arbitrary number of evenly spaced bids, as the number of acceptable bids is increased, the expected revenue of the seller and the probability of ex post inefficiency both may either increase or decrease.

Keywords: Auctions/Bidding; Auction Design; Discrete Bidding.

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1 Introduction

The existing literature on auctions focuses almost exclusively on situations in which any bid on a continuous interval can be submitted. In practice bidders are not able to choose their bid from a continuum of acceptable points. At the most basic level, the discrete nature of currency imposes a restriction on the acceptable bid levels. Additionally, the set of acceptable bids may be further restricted by the auctioneer. In this paper we analyze a sealed bid second price auction, in which acceptable bids are restricted to a set of discrete values.

In practice, the bid space is restricted in a discrete manner on most internet auction sites by the imposition of a positive bid increment. On eBay, this bid increment varies between five cents and one hundred dollars, depending upon the level of the current high bid. It should also be noted that most internet auctions are conducted in such a manner so as to make them strategically equivalent to second price auctions.[1]

Auctions in which the bid space is restricted in a discrete manner have been analyzed by Chwe (1989), Rothkopf and Harstad (1994), and Yu (1999). Chwe considers a sealed bid first price auction with discrete bidding. Specifically, he considers a situation in which there are n bidders, each with an independent private valuation drawn from the cumulative distribution function $F(v) = v$. The unique feature of his model is that bidders are not allowed to submit any bid on the interval $[0, \infty)$, but rather are restricted to choosing among M "evenly spaced" discrete bid levels $\{b_1, b_2, \ldots, b_M\}$ such that $b_i = \frac{i-1}{M}$. Since submitting a bid above ones valuation is a dominated strategy in a first price auction, Chwe is essentially allowing there to be an additional bid of $b_{M+1} = \frac{(M+1)-1}{M} = 1$. That is, he implicitly assumes that the highest acceptable bid is exactly equal to the highest possible bidder valuation. Chwe characterizes a unique symmetric pure strategy equilibrium for this auction. He subsequently argues that such an auction results in less revenue for the seller than the traditional continuous first price sealed bid auction. Further, as $M \to \infty$, the equilibrium bids converge to the equilibrium bids in the continuous case. Thus, the revenue of the seller in this discrete auction converges to the revenue of the continuous auction as $M \to \infty$. However, it is shown by way of example that the expected revenue of the seller is not monotonic in the number of acceptable bids.

Rothkopf and Harstad analyze an English auction in which bids are restricted to discrete values. It is important to note that (for bidders with independent, private valuations) an English auction with discrete bidding is *not* strategically equivalent to a sealed bid second price auction with discrete bidding, in contrast to the corresponding auctions with continuous bidding. To recognize this, first note that in an English auction submitting a bid above one's true valuation is dominated by submitting a bid at or below one's true valuation, be it an English auction with discrete or continuous bidding. Likewise, in a sealed bid second price auction with continuous bidding, submitting a bid above one's true valuation is dominated by submitting a bid equal to one's true valuation. However, as will be argued below, in a sealed bid second price auction with discrete bidding, submitting a bid above one's true valuation is not immediately dominated.

In the English auction analyzed by Rothkopf and Harstad n bidders (each with an independent, private valuation) are restricted to using $m + 1$ discrete bids, with the highest bid level exactly equal to the highest possible bidder valuation. Under numerous assumption on bidder

[1] For example, under the "proxy bidding" system on eBay (in which the highest bid competes at the minimum level necessary to be the leading bid), a bidder with an independent private valuation has a dominant strategy of submitting a bid equal to his true valuation, a point which has been recognized by Lucking-Reiley (2000a and 2000b).

behavior, expressions are determined for "expected seller revenue," "expected lost revenue" (the expected difference between the second highest bidder valuation and realized auction revenue), and "expected economic inefficiency" (the expected difference between the highest bidder valuation and the valuation of the winning bidder). When bidder valuations are uniformly distributed, expected seller revenue is maximized by bid levels which minimize expected inefficiency (this is not necessarily true if valuations are not uniformly distributed). Further, expected seller revenue may be maximized by having either an increasing, decreasing, or constant distance between subsequent bids (the result depending upon the number of bidders and distribution from which bidder valuations are drawn).

Yu examines each of the four common auction forms (sealed bid first price, sealed bid second price, English auction, and Dutch auction) under the assumption that bids are restricted to "evenly spaced" discrete values. The valuations of the n bidders are assumed to be independently drawn from a common distribution $F(v)$ such that $F(0) = 0$ and $F(1) = 1$. Yu assumes that bidders are restricted to choosing from $M + 1$ acceptable bid levels $\{b_1, b_2, \ldots, b_M, b_{M+1}\}$ such that $b_i = \frac{i-1}{M}$. Thus, she explicitly assumes that the highest acceptable bid level is exactly equal to the highest possible bidder valuation. It is shown that in each such auction a symmetric pure strategy equilibrium exists.[2] However, the sealed bid second price auction is no longer dominance solvable. Additionally, for the sealed bid second price auction, in equilibrium some bidders will bid above their valuation and some bidders will bid below their valuation. Finally, for each type of auction, as $M \to \infty$ equilibrium bids converge to the equilibrium bids in the corresponding continuous bid auction.

Before we look into the contributions made by this paper, it may be worthwhile to briefly mention of another vein in the literature. A few papers have looked into the phenomenon of "jump bidding" in ascending English auctions wherein bidders sometimes bid higher than what is necessary to be the current highest bidder. Isaac et.al.(2007), Avery (1998), Easley and Tenorio (2004) are few of the notable ones. As shall be subsequently seen, papers such as above are examining an auction format different from ours (open out-cry versus sealed bid) which do not always operate in the IPV domain. However we do feel obliged to mention them here, as the auctions described in them also demonstrate a discrete bid space similar to ours.

Turning to this paper, we consider a situation in which the discrete bid points need not be "evenly spaced," an assumption that was made by both Chwe and Yu.[3] Additionally, we do not assume that the highest acceptable bid point must be exactly equal to the highest possible bidder valuation, an assumption that was explicitly made by both Rothkopf and Harstad and Yu and implicitly made by Chwe.[4] A central point of our analysis is that these previous assumptions were quite restrictive. This point is made by illustrating that a revenue maximizing seller: may wish to set a high bid level which is not equal to the highest possible bidder valuation, and may wish to have discrete bids that are not evenly spaced. Throughout the present analysis, attention is restricted to a situation in which there are two bidders, each with an independent private valuation

[2] Yu does not show uniqueness of a symmetric pure strategy equilibrium.

[3] When reference is made to Chwe, it is simply done to note that the strategy space in the auction we are analyzing is different from the strategy space in the auction analyzed by Chwe. Since Chwe considered a first price auction whereas we are considering a second price auction, the payoff functions for bidders clearly differ. Thus, the equilibrium which we identify would not extend to a first price auction as examined by Chwe.

[4] Even though bidders in a sealed bid first price auction with discrete bidding (as analyzed by Chwe) would never submit a bid equal to the highest possible bidder valuation, making this assumption is restrictive when made in conjunction with the assumption of evenly spaced bids.

drawn from a common distribution. An examination of bidder behavior in such an environment is presented in Section 3. Restricting attention to symmetric pure strategy behavior on the part of bidders, a unique equilibrium is shown to exist. When bidders play according to this unique pair of strategies, the item may be awarded to the bidder with the lower valuation. That is, there is a strictly positive probability of allocative inefficiency.

Seller revenue is subsequently examined in Section 4. In order to demonstrate that the assumptions on the placement of the discrete bid points made by Chwe, Rothkopf and Harstad, and Yu were restrictive, we begin by considering situations in which there are only two acceptable bids (subsection 4.1) or three acceptable bids (subsection 4.2). When constrained to choosing two bid levels, a revenue maximizing seller may choose a level of the "highest acceptable bid" which is greater than or less than the highest possible valuation. In this case, such a revenue maximizing seller does not necessarily minimize the probability of allocative inefficiency. Additionally, a revenue maximizing seller constrained to choosing three discrete bid values may wish to set unevenly spaced bids. Finally, a numerical analysis of an auction with an arbitrary number of evenly spaced bids is conducted (subection 4.3), in order to examine how the equilibrium outcome changes as the distance between evenly spaced bids is reduced. As more discrete bid points below the highest possible bidder valuation become available (by way of the common distance between bids being reduced), both the expected revenue of the seller and the probability of ex post inefficiency may either increase or decrease. That is, reducing the spacing between discrete bid points will not always increase the expected revenue of the seller and will not always decrease the probability of ex post inefficiency. Section 5 concludes.

2 Model

A single item is for sale by way of auction. There are two bidders, each with an independent private valuation, $v_i \in [v_L, v_H]$, drawn from a common continuous distribution $F(v)$, such that: $F(v_L) = 0$, $F(v_H) = 1$, and $f(v) = F'(v) > 0$ for all $v \in [v_L, v_H]$. Each bidder i simultaneously submits a sealed bid b_i. However, bids are constrained to a countable number of discrete points $B_0 < B_1 < \ldots < B_j < \ldots$. It is assumed throughout that $B_0 = v_L$.[5]

Once both bids have been submitted, the seller (weakly) orders the bids. If both bidders submit the same bid B_j, the bids are ordered randomly by the seller (with each ordering occurring with equal probability). The item is awarded to the bidder submitting the highest bid for an amount equal to the second highest bid submitted.[6]

It is well known that in a standard sealed bid second price auction with continuous bidding, bidders with independent private valuations have a dominant strategy of submitting a bid equal to their own valuation (Vickrey (1961)). However, if bids are constrained to a discrete set of points, a bidder may be unable to submit such a bid. This is precisely the case here, when valuations are drawn from a continuous distribution. It is not immediately clear what a bidder should do in such an environment. The first goal is to characterize optimal bidder behavior. Once this is done, the expected revenue of the seller is analyzed. An attempt is made to determine how the revenue of the seller changes as the discrete bid levels change and as additional bid levels are allowed.

[5]This is analogous to considering a traditional second price auction without a reserve price.

[6]For example, suppose the two bidders submit the following bids: $b_1 = B_2$ and $b_2 = B_2$. The bids will be order by the seller as either $b_1 \geq b_2$ or $b_2 \geq b_1$, each with equal probability. In either case, the second highest bid is equal to B_2. Thus, the item will be sold at a price of B_2 to either bidder 1 or bidder 2.

3 Bidder Behavior

Suppose there are an arbitrary number of discrete bid points, a finite number of which are less than v_H. If there are no acceptable bid levels greater than v_H, let B_k denote the largest acceptable bid level (clearly $B_k \leq v_H$ in this case). If there is at least one acceptable bid level (weakly) greater than v_H, let B_{k-1} denote the largest acceptable bid level (strictly) less than v_H and let B_k denote the smallest acceptable bid level (weakly) greater than v_H.[7]

3.1 Dominated Bidding Strategies

Consider a bidder with valuation v_i. Proposition 1 serves as an initial characterization of the behavior of such a bidder. The stated behavior is analogous to the weakly dominant strategy of "truthful bidding" in a traditional sealed bid second price auction. All results are proved in Appendix A.

Proposition 1 *Consider an arbitrary allowable bid level B_j. For a bidder with $v_i \leq B_j$, bidding B_j weakly dominates bidding above B_j; for a bidder with $v_i \geq B_j$, bidding B_j weakly dominates bidding below B_j.*

Considering a bidder with $v_i \in (B_{j-1}, B_j)$, Proposition 1 implies that all bids other than B_{j-1} and B_j are weakly dominated.[8] As in the proof of Proposition 1, define p_j to be the probability with which a rival bidder submits a bid of B_j. The following results follow immediately from Proposition 1 (and are thus stated without proof).

Corollary 1 *If $p_j > 0$, then a bidder with $v_i \geq B_j$ realizes a strictly higher payoff from bidding B_j than by bidding below B_j.*

Corollary 2 *If $p_j > 0$, then a bidder with $v_i \leq B_j$ realizes a strictly higher payoff from bidding B_j than by bidding above B_j.*

When characterizing bidder behavior, the search for an equilibrium will be restricted to symmetric, pure strategy bidding functions. It will be argued that within this restricted class of strategies, a unique equilibrium exists.

3.2 Symmetric Equilibrium Bidding Strategies

Focus on the payoff of a bidder with $v_i \in (B_{j-1}, B_j)$ from bidding either B_{j-1} or B_j (with the behavior of his rival fixed). The expected payoff of i from bidding B_j is

$$\pi_i(B_j, v_i) = \sum_{l=0}^{j-1}(v_i - B_l)p_l + (v_i - B_j)\frac{1}{2}p_j \tag{1}$$

[7] As will be seen, this seemingly inconsistent description of the bid levels will result in bidder i choosing from exactly $k+1$ bids in equilibrium in either case.

[8] Similarly, if the maximum acceptable bid is less than v_H, all bids other than B_k (the largest acceptable bid in this case) are weakly dominated for a bidder with $v_i \geq B_k$.

The difference between the expected payoff from bidding B_j versus B_{j-1} is

$$\pi_i(B_j, v_i) - \pi_i(B_{j-1}, v_i) = (v_i - B_j)\frac{1}{2}p_j + (v_i - B_{j-1})\frac{1}{2}p_{j-1}. \tag{2}$$

Given the results thus far, we establish the following:

Lemma 1 *For any pure strategy symmetric equilibrium, it must be that* $p_j > 0$ *for* $j = 0, 1, 2, \ldots, k$ *and* $\sum_{j=0}^{k} p_j = 1$.

Note that when describing a symmetric equilibrium, p_j refers to the probability with which each individual bidder chooses to submit the bid B_j. Lemma 1 implies that in any such equilibrium, all bids from B_0 up to B_k are submitted with positive probability (while bids of B_{k+1} and above are submitted with zero probability). Also observe that $\pi_i(B_j, v_i)$ is linear in v_i. Further, using equation (2) it can be easily seen that

$$\pi_i(B_j, v_L) < \pi_i(B_{j-1}, v_L). \tag{3}$$

Moreover,

$$\frac{\partial \pi_i(B_{j-1}, v_i)}{\partial v_i} < \frac{\partial \pi_i(B_j, v_i)}{\partial v_i}. \tag{4}$$

Consider $j = 1, 2, \ldots, k-1$. First substituting $v_i = B_j$ in (2) and then again $v_i = B_{j-1}$ in the same equation, we get the following respectively.

$$\pi_i(B_j, B_j) > \pi_i(B_{j-1}, B_j) \tag{5}$$

$$\pi_i(B_j, B_{j-1}) < \pi_i(B_{j-1}, B_{j-1}) \tag{6}$$

The above inequalities (3)–(6) imply that for any bidder i, $\pi_i(B_j, v_i)$ intersects $\pi_i(B_{j-1}, v_i)$ *from below* and that there exists an unique $c_j \in (B_{j-1}, B_j)$ at which both are equal.[9] Linearity of expected profit with respect to v_i ensures that the two functions intersect only once.

Finally, (5) and (6) also holds for $j = k$ implying that there exists an unique $c_k \in (B_{k-1}, B_k)$ at which expected profit functions are equal. In fact, it can be easily seen that $c_k \in (B_{k-1}, v_H)$. This is quite obvious if $B_k \leq v_H$. In case of $B_k > v_H$, this is ensured by Lemma 1. As $p_k > 0$, we must have $\pi_i(B_k, v_H) > \pi_i(B_{k-1}, v_H)$, otherwise p_k can never be positive in equilibrium.

Recall that for a bidder with $v_i \in (B_{j-1}, B_j)$, Proposition 1 implies that all bids other than B_{j-1} and B_j are weakly dominated. Focusing on B_j (the higher of these two remaining non-dominated bid points, which is greater than the valuation of the bidder) bidder i will consider submitting a bid of B_j in a second price environment precisely because doing so does not guarantee that he will have to pay an amount above his valuation. For this bidder, bidding B_j increases the likelihood of winning the auction, but at the risk of having to pay an amount above his valuation. A rational bidder will balance this benefit of bidding B_j (the greater likelihood of winning the auction) against this cost of bidding B_j (the possibility of having to pay an amount above his valuation).

For a bidder with $v_i \in (B_{j-1}, B_j)$, bidding B_{j-1} gives a strictly higher payoff *if and only if* his valuation is above a cutoff $c_j \in [B_{j-1}, B_j]$ while bidding B_j gives a strictly higher payoff *if*

[9] Note that these inequalities hold strictly (for all bids B_{j-1} and B_j) only if all bids are submitted with strictly positive probability. As stated by Lemma 1, this must be the case for any pure strategy symmetric equilibrium.

and only if his valuation is below c_j. Figure 1 in Appendix B illustrates this phenomenon for a bidder i facing a choice between bidding B_{j-1} and B_j.

Consider the bid points $B_0 < B_1 < \ldots < B_{k-1} < B_k$. For any such $k+1$ acceptable bid points, conjecture that there exists a unique set of $k+2$ values $c_0 < c_1 < \ldots < c_k < c_{k+1}$ such that $c_0 = v_L$, $c_j \in (B_{j-1}, B_j)$ for $j = 1, \ldots, k$, and $c_{k+1} = v_H$, with the interpretation that in equilibrium a bidder with valuation v_i will choose $b_i = B_j$ if and only if $v_i \in [c_j, c_{j+1})$ for $j = 0, 1, \ldots, k-1$ and choose $b_i = B_k$ if and only if $v_i \in [c_k, c_{k+1}]$.

If the rival of bidder i plays according to such a strategy characterized by $C = (c_0, c_1, \ldots, c_k, c_{k+1})$, bidder i with valuation v_i has an expected payoff of

$$\pi_i(B_0, v_i, C) = (v_i - B_0)\frac{1}{2}F(c_1)$$

from submitting a bid of $b_i = B_0$ and an expected payoff of

$$\pi_i(B_j, v_i, C) = \left\{\sum_{l=0}^{j-1}(v_i - B_l)\left[F(c_{l+1}) - F(c_l)\right]\right\} + (v_i - B_j)\frac{1}{2}\left[F(c_{j+1}) - F(c_j)\right]$$

from submitting a bid of $b_i = B_j$ for $j = 1, \ldots, k$.

In order for the conjectured values of $c_1 < \ldots < c_k$ to support an equilibrium, it must be that $\pi_i(B_{j-1}, c_j, C) = \pi_i(B_j, c_j, C)$, or equivalently $D_j(c_{j-1}, c_j, c_{j+1}) = \pi_i(B_{j-1}, c_j, C) - \pi_i(B_j, c_j, C) = 0$, for $j = 1, \ldots, k$. This leads to k conditions that must simultaneously be satisfied by the k values $c_1 < \ldots < c_k$. These k conditions can be expressed as

$$(B_j - c_j)\left[F(c_{j+1}) - F(c_j)\right] - (c_j - B_{j-1})\left[F(c_j) - F(c_{j-1})\right] = 0 \tag{7}$$

for $j = 1, \ldots, k$.[10] Theorem 1 characterizes equilibrium bidder behavior.

Theorem 1 *A symmetric equilibrium exists in which each bidder i submits a bid of $b_i = B_j$ if and only if $v_i \in [c_j, c_{j+1})$ for $j = 0, \ldots, k-1$ and submits a bid of $b_i = B_k$ if and only if $v_i \in [c_k, c_{k+1}]$, where $c_0 = v_L$, $c_{k+1} = v_H$, and $(c_1, \ldots, c_k)$ are the unique values for which condition (7) is simultaneously satisfied for $j = 1, \ldots, k$.*

The unique values of $c_1 < \ldots < c_k$ for which condition (7) holds simultaneously for $j = 1, \ldots, k$ specify equilibrium bidder behavior in this auction. It is important to take note of two important characteristics of the equilibrium characterized in Theorem 1. First, it is unique in the class of symmetric pure strategy equilibria (follows from our discussion preceding the statement of the theorem above). And secondly, this equilibrium is dominance solvable to the extent of narrowing down to two bid points which are *nearest* to the valuation of any bidder.[11]

Figure 2 in Appendix B directly illustrates the equilibrium expected payoff for a bidder i from each allowable bid, in the case of four allowable bid points. As can be seen readily, for any $v_i \in (c_j, c_{j+1})$, bidding B_j fetches the highest payoff for bidder i, where $j = 0, 1, 2, 3$.

When following such a strategy, a bidder may choose to submit a bid exceeding his valuation. Recall that a bidder with $v_i \in (B_{j-1}, B_j)$ chooses to bid either B_{j-1} or B_j based upon a careful comparison of the benefit of bidding above his valuation (i.e., the increased likelihood of winning

[10] Recall that $c_0 = v_L$ and $c_{k+1} = v_H$. As a result, $F(c_0) = 0$ and $F(c_{k+1}) = 1$.

[11] Both characteristics have a nice resemblance to the case of standard second price auctions with a continuous bid space.

the auction when bidding B_j) to the cost of bidding above his valuation (i.e., the possibility of having to pay B_j, an amount greater than v_i). For a bidder with a valuation relatively close to B_j, the loss of having to pay a price above v_i is smaller than for a bidder with a valuation relatively close to B_{j-1}. Thus, for each $j = 1, \ldots, k$ the bidders with "relatively high valuations" within each range (specifically, $v_i \in [c_j, B_j)$) will choose to bid above their valuation, precisely because the gain from doing so exceeds the cost from doing so. Likewise, those bidders with "relatively low valuations" within each range (specifically, $v_i \in [B_{j-1}, c_j)$) will choose to not bid above their valuation, precisely because the cost from doing so exceeds the gain from doing so.

Further, note that the item is awarded to the bidder with the lower valuation with strictly positive probability. Specifically, whenever both bidders submit the same bid, the allocation of the item is inefficient with probability $\frac{1}{2}$. Thus, the total probability of such allocative inefficiency is

$$\Pr(I) = \sum_{j=0}^{k} \frac{1}{2} \left[F(c_{j+1}) - F(c_j)\right]^2 .$$

4 Seller Revenue

As with a traditional sealed bid second price auction, the revenue of the seller is equal to the second highest bid submitted. Therefore, the expected revenue of the seller depends critically upon the distribution of the second highest valuation of the two bidders, denoted $F_{(1)}(v) = 2F(v) - F(v)^2$. The expected revenue of the seller can be expressed as

$$\begin{aligned}
\Pi_S &= B_0 F_{(1)}(c_1) + B_1[F_{(1)}(c_2) - F_{(1)}(c_1)] + \cdots \\
&\quad + B_{k-1}[F_{(1)}(c_k) - F_{(1)}(c_{k-1})] + B_k[1 - F_{(1)}(c_k)] \\
&= \sum_{j=0}^{k} B_j[F_{(1)}(c_{j+1}) - F_{(1)}(c_j)] \\
&= B_k - \sum_{j=1}^{k} (B_j - B_{j-1}) F_{(1)}(c_j).
\end{aligned}$$

To demonstrate that the assumptions on the placement of the discrete bid points made in previous studies were in fact restrictive, we begin our examination of seller revenue by considering situations in which there are only two or only three acceptable bids. With only two bid levels, a revenue maximizing seller will not necessarily want to choose a bid exactly equal to the highest possible bidder valuation (a restriction on the highest acceptable bid made by Chwe, Rothkopf and Harstad, and Yu). When choosing the levels of three discrete bids, evenly spaced bids (a restriction on bid placement made by both Chwe and Yu) are not necessarily revenue maximizing. Finally, such an auction with an arbitrary number of evenly spaced bids (the highest of which need not be equal to the highest possible bidder valuation) is analyzed, in order to gain further insight into the outcome as the number of relevant bids and the distance between bids is changed.

4.1 Two Discrete Bids

Consider a situation in which there are only two acceptable bids, $B_0 = 0$ and $B_1 > 0$. When able to choose the value of B_1, an expected revenue maximizing seller will not always want to

set $B_1 = v_H$ and will not always choose the value of B_1 which minimizes the probability of ex post inefficiency. To illustrate these points, consider a distribution function $F(\cdot)$ satisfying the additional assumptions that: $v_L = 0$; and $x\frac{f(x)}{1-F(x)}$ is strictly increasing in x. First note that for a general distribution function $F(\cdot)$, equilibrium bidder behavior defined by Condition (7) can be expressed as:

$$B_1\left[1 - F(c_1)\right] - c_1 = 0. \tag{8}$$

Letting $c_1(B_1)$ denote the value of c_1 for which this condition is satisfied, by the Implicit Function Theorem we have $c_1'(B_1) = \frac{1-F(c_1)}{1+B_1 f(c_1)} > 0$. That is, the value of c_1 is increasing in B_1. Therefore, there are two distinct effects on the expected revenue of the seller from a change in the value of B_1. An increase in B_1 will: increase expected revenue, since the seller will receive a greater amount when both bidders have valuations above c_1; but decrease expected revenue, since it becomes less likely that both bidders will have valuations above c_1. When choosing the value of B_1, an expected revenue maximizing seller will balance these two effects against one another. With only two discrete bids (and $B_0 = 0$), the expected revenue of the seller as a function of B_1 can be expressed as:

$$\begin{aligned}\Pi_S(B_1) &= B_1\left[1 - F_{(1)}\left(c_1(B_1)\right)\right] \\ &= B_1\left[1 - 2F\left(c_1(B_1)\right) + F\left(c_1(B_1)\right)^2\right].\end{aligned}$$

From here:

$$\Pi_S'(B_1) = \left[1 - F\left(c_1(B_1)\right)\right]^2 - 2B_1 f\left(c_1(B_1)\right) c_1'(B_1)\left[1 - F\left(c_1(B_1)\right)\right].$$

For an increase in B_1: the first term in the expression above captures the increase in expected revenue resulting from receiving a larger payment when both bidders bid B_1; the second term captures the decrease in expected revenue from having it be less likely that both bidders will submit a bid of B_1. Since $c_1'(B_1) = \frac{1-F(c_1)}{1+B_1 f(c_1)}$, this expression can be simplified to:

$$\Pi_S'(B_1) = \left[1 - F\left(c_1(B_1)\right)\right]^2 \left\{1 - 2B_1 \frac{f\left(c_1(B_1)\right)}{1 + B_1 f\left(c_1(B_1)\right)}\right\},$$

the sign of which is the same as the sign of $1 - B_1 f\left(c_1(B_1)\right)$. By Condition (8) the relation between B_1 and c_1 can be stated as $B_1 = \frac{c_1}{1-F(c_1)}$. Thus, the sign of $\Pi_S'(B_1)$ is the same as the sign of $1 - c_1\frac{f(c_1)}{1-F(c_1)}$. $1 - c_1\frac{f(c_1)}{1-F(c_1)}$ is equal to one for $c_1 = v_L = 0$ and tends to $-\infty$ as $c_1 \to v_H$. Further, under the assumption that $x\frac{f(x)}{1-F(x)}$ is increasing in x, $1 - c_1\frac{f(c_1)}{1-F(c_1)}$ is strictly decreasing in c_1 for $c_1 \in (0, v_H)$. Therefore, there exists a unique value $c_1^* \in (0, v_H)$ such that $1 - c_1\frac{f(c_1)}{1-F(c_1)}$ is equal to zero for $c_1 = c_1^*$, strictly positive for $c_1 < c_1^*$, and strictly negative for $c_1 > c_1^*$. It follows that there exists a unique B_1^* such that: $\Pi_S'(B_1^*) = 0$, $\Pi_S'(B_1) > 0$ for $B_1 < B_1^*$, and $\Pi_S'(B_1) < 0$ for $B_1 > B_1^*$. c_1^* is the unique value of c_1 and B_1^* is the unique value of B_1 for which the seller's expected revenue is maximized. Turning attention to the resulting probability of ex post inefficiency, when there are only two discrete bid points:

$$\Pr(I) = \frac{1}{2}\left(1 - 2F(c_1) + 2F(c_1)^2\right).$$

From this expression, $\frac{d\Pr(I)}{dc_1} = -f(c_1)\left[1 - 2F(c_1)\right]$, implying that the value of c_1 which minimizes the probability of ex post inefficiency is the unique value c_1^E for which $F\left(c_1^E\right) = \frac{1}{2}$. Let

B_1^E denote the corresponding value of B_1 which minimizes $\Pr(I)$. The observation that c_1^E must satisfy $F\left(c_1^E\right) = \frac{1}{2}$ is rather intuitive. Inefficiency will result half of the time whenever both bidders have valuations on the same side of c_1. Therefore, (irrespective of the functional form of $F(\cdot)$) this probability of inefficiency is minimized by minimizing the likelihood that both bidder valuations will be on the same side of c_1. This is done only by setting a value of c_1 for which it is equally likely that any one bidder will have a valuation either above or below c_1: that is, a value such that $F\left(c_1^E\right) = \frac{1}{2}$. A final implication is that regardless of the functional form of $F(\cdot)$, the minimum probability of allocative inefficiency with only two discrete bid points is $\Pr(I)^E = \frac{1}{4}$. A comparison of B_1^* to v_H will allow us to see if an expected revenue maximizing seller would set B_1 equal to the highest possible bidder valuation, while a comparison of B_1^* to B_1^E (or of c_1^* to c_1^E) will allow us to determine if an expected revenue maximizing seller would choose B_1 to minimize the probability of ex post inefficiency. In general the seller will not necessarily want to set $B_1^* = v_H$ or $B_1^* = B_1^E$. To see this, consider $F(v) = v^\alpha$, with $\alpha \in (0, \infty)$. For this distribution function, $1 - c_1 \frac{f(c_1)}{1-F(c_1)} = 1 - \frac{\alpha c_1^\alpha}{1-c_1^\alpha}$. It follows that the values of c_1^* and B_1^* can be expressed as functions of α as:

$$c_1^*(\alpha) = \left(\frac{1}{1+\alpha}\right)^{\frac{1}{\alpha}} \qquad \text{and} \qquad B_1^*(\alpha) = \frac{(1+\alpha)^{\frac{\alpha-1}{\alpha}}}{\alpha}.$$

Proposition 2 characterizes $B_1^*(\alpha)$ in relation to $v_H = 1$, the highest possible bidder valuation.

Proposition 2 *$B_1^*(1) = 1$; $B_1^*(\alpha) > 1$ for $\alpha < 1$; and $B_1^*(\alpha) < 1$ for $\alpha > 1$.*

If bidder valuations are uniformly distributed between zero and one ($\alpha = 1$), then $B_1^*(\alpha) = 1 = v_H$. However, if bidder valuations are not uniformly distributed ($\alpha \neq 1$), then $B_1^*(\alpha) \neq 1 = v_H$. When interpreting the outcome for $\alpha \neq 1$, start by noting that when the seller sets a bid level B_1^* leading to c_1^*, the probability that any one bidder will choose to submit a bid of B_1^* is $1 - F\left(c_1^*\right) = \frac{\alpha}{1+\alpha}$. Since this probability is increasing in α, it follows that from the perspective of any one bidder it is: less likely that his rival will submit a bid of B_1^* when the value of α is smaller; and more likely that his rival will submit a bid of B_1^* when the value of α is larger. If $\alpha < 1$ (in which case bidder valuations are drawn from a distribution which is First Order Stochastically Dominated by the distribution function from which a uniformly distributed random variable is drawn), the expected revenue of the seller is maximized by a bid level strictly above the highest possible bidder valuation. In this case, any bidder is more willing to bid B_1^* precisely because it is less likely that he will actually have to pay B_1^* (since $1 - F\left(c_1^*\right)$ is smaller when α is smaller). Thus, the gain to the seller from choosing a relatively high value of B_1 outweighs the loss from doing so, implying that $B_1^* > 1$ is optimal. When $\alpha > 1$ (in which case bidder valuations are drawn from a distribution which First Order Stochastically Dominates the distribution function from which a uniformly distributed random variable is drawn), the expected revenue of the seller is maximized by a bid level strictly below the highest possible bidder valuation. In this case, any bidder is now less willing to bid B_1^* because he is more likely to have to pay B_1^* when doing so (since $1 - F\left(c_1^*\right)$ is larger when α is larger). As a result, the gain to the seller from choosing a higher value of B_1 is outweighed by the loss from choosing a higher value of B_1, so that $B_1^* < 1$ is best. In summary, the assumption in the existing literature that there must be a bid level exactly equal to the highest possible bidder valuation is restrictive, since an expected revenue maximizing seller (constrained to choosing a specific number of discrete bid points) will generally not want to have an acceptable bid level equal to the highest possible bidder valuation. Shifting focus to

the probability of allocative inefficiency, note that when $F(v) = v^\alpha$ it follows that c_1^E and B_1^E can be expressed as functions of α as:

$$c_1^E(\alpha) = \left(\frac{1}{2}\right)^{\frac{1}{\alpha}} \qquad \text{and} \qquad B_1^E(\alpha) = 2^{\frac{\alpha-1}{\alpha}}.$$

Proposition 3 characterizes $B_1^E(\alpha)$ in relation to $v_H = 1$, the highest possible bidder valuation.

Proposition 3 *$B_1^E(1) = 1$; $B_1^E(\alpha) < 1$ for $\alpha < 1$; and $B_1^E(\alpha) > 1$ for $\alpha > 1$.*

Together Propositions 2 and 3 allow for a straightforward comparison of the value of B_1 that maximizes the expected revenue of the seller (that is, $B_1^*(\alpha)$) to the value of B_1 that minimizes the probability of allocative inefficiency (that is, $B_1^E(\alpha)$).

When bidder valuations are uniformly distributed between zero and one ($\alpha = 1$) we have $B_1^*(1) = B_1^E(1) = 1$, implying that the probability of ex post inefficiency is minimized when the expected revenue of the seller is maximized. However, if bidder valuations are not uniformly distributed ($\alpha \neq 1$), then $B_1^*(\alpha) \neq B_1^E(\alpha)$. When $\alpha > 1$ (in which case bidder valuations are drawn from a distribution which First Order Stochastically Dominates the distribution function from which a uniformly distributed random variable is drawn), we have $B_1^*(\alpha) < 1 < B_1^E(\alpha)$. That is, the expected revenue of the seller is maximized by a value of B_1 strictly less than the value which minimizes the probability of allocative inefficiency. Finally, if $\alpha < 1$ (in which case bidder valuations are drawn from a distribution which is First Order Stochastically Dominated by the distribution function from which a uniformly distributed random variable is drawn), we observe $B_1^E(\alpha) < 1 < B_1^*(\alpha)$. In this case, the expected revenue of the seller is maximized by a value of B_1 strictly greater than the value which minimizes the probability of ex post inefficiency. In summary, for $F(v) = v^\alpha$, the expected revenue maximizing value of B_1 minimizes the probability of inefficiency only when bidder valuations are uniformly distributed.

Recall that when $\alpha = 1$, the expected revenue of the seller is maximized by $B_1^* = 1$. In this case, the resulting expected revenue of the seller is $\Pi_S(B_1^*) = \frac{1}{4}$, which is strictly less than the expected revenue of $\Pi_T^* = \frac{1}{3}$ from a traditional sealed bid second price auction in which any bid on the interval $[0, 1]$ could be submitted. However, for $\alpha \neq 1$, the seller may be able to realize a higher expected revenue from a sealed bid second price auction with only two discrete bid levels than from a traditional sealed bid second price auction in which any positive bid can be submitted.[12] To see this, begin by noting that with two discrete bid levels, the maximum expected revenue of the seller as a function of α is equal to

$$\Pi_S^*(\alpha) = B_1^*(\alpha)\left\{1 - F\left(c_1^*(\alpha)\right)\right\}^2 = \frac{\alpha}{(1+\alpha)^{\frac{\alpha+1}{\alpha}}}.$$

The expected revenue for the seller from a traditional second price auction with continuous bidding (and no reserve price) can be expressed as a function of α as

$$\Pi_T(\alpha) = \int_0^1 v\left(2\alpha v^{\alpha-1} - 2\alpha v^{2\alpha-1}\right) dv = \frac{2\alpha^2}{(1+\alpha)(2\alpha+1)}.$$

[12] Clearly the expected revenue from the discrete second price auction considered here cannot exceed that from the optimal auction characterized by Myerson (1981). However, a second price auction with a continuous bid space and no reserve price provides an appropriate benchmark for comparison, since in the discrete auction considered here (like those analyzed by Chwe (1989), Rothkopf and Harstad (1994), and Yu (1999)) the lowest acceptable bid is equal to the lowest possible bidder valuation.

The seller is able to realize a greater expected revenue from a discrete auction with only two bid levels so long as $\Pi_S^*(\alpha) > \Pi_T(\alpha)$, or equivalently

$$2\alpha\left[1-(1+\alpha)^{\frac{1}{\alpha}}\right]+1>0.$$

This condition is not satisfied for $\alpha = 1$, verifying the observation that when facing bidders with uniformly distributed valuations, the seller is able to realize a higher expected payoff from an analogous traditional auction with continuous bidding. For $\alpha \neq 1$ this inequality may or may not hold. It is straightforward to verify that this inequality is not satisfied for $\alpha = 2$ and $\alpha = \frac{1}{2}$, but is satisfied for $\alpha = \frac{1}{4}$ and $\alpha = \frac{1}{10}$.[13] Thus, it appears as if for α sufficiently low the seller realizes a higher expected revenue from a second price auction with only two discrete bids than from a traditional second price auction with continuous bidding.

4.2 Three Discrete Bids and Inter-Bid Spacing

Consider a discrete auction with three bids: $B_0 = 0$, B_1, and B_2. Suppose $F(v) = v$, so that bidder valuations are uniformly distributed. From subsection 4.1, we have that $B_1^* = 1$ maximizes the expected revenue of the seller when there are only two allowable bids. In the situation when there are three allowable bids, it can easily be seen that the seller has no incentive to have $B_1 > 1$.[14] Thus, when focusing on the expected revenue of the seller in an auction with three discrete bids, attention can be restricted to $B_1 \leq 1$.

In an auction with three discrete bids, the equilibrium behavior of bidders with uniformly distributed valuations is dictated by the unique values of c_1 and c_2 satisfying the following conditions:

$$(B_1 - c_1)(c_2 - c_1) - c_1^2 = 0$$

and

$$(B_2 - c_2)(1 - c_2) - (c_2 - B_1)(c_2 - c_1) = 0.$$

Let $c_1(B_1, B_2)$ and $c_2(B_1, B_2)$ denote these values. The expected revenue of the seller in this case can be expressed as:

$$\begin{aligned}\Pi_S(B_1,B_2) &= B_2 - B_1\left[2c_1(B_1,B_2) - c_1(B_1,B_2)^2\right] \\ &\quad -(B_2-B_1)\left[2c_2(B_1,B_2)-c_2(B_1,B_2)^2\right].\end{aligned}$$

Constraining $B_2 = 2B_1$, we have $\widehat{c}_1(B_1) = c_1(B_1, 2B_1)$ and $\widehat{c}_2(B_1) = c_2(B_1, 2B_1)$. Such evenly spaced bids lead to an expected revenue

$$\begin{aligned}\widehat{\Pi}_S(B_1) &= \Pi_S(B_1, 2B_1) \\ &= B_1\left\{2-\left[2\widehat{c}_1(B_1)-\widehat{c}_1(B_1)^2\right]-\left[2\widehat{c}_2(B_1)-\widehat{c}_2(B_1)^2\right]\right\}.\end{aligned}$$

Based upon a numerical analysis, $\widehat{\Pi}_S(B_1)$ is maximized by $\widehat{B}_1^* \approx .45528513$. Since the two bid points are constrained to be evenly spaced, this leads to $\widehat{B}_2^* \approx .91057027$. For these bid levels, $\widehat{c}_1 \approx .27018243$ and $\widehat{c}_2 \approx .66455019$, resulting in $\widehat{\Pi}_S^* \approx .29373188$.

[13] Numerical calculations suggest that this inequality holds if and only if $\alpha < \bar{\alpha}$, with $\bar{\alpha} \approx .37341058$.

[14] If $B_1 \geq 1$, then bidding B_1 dominates B_2 for each bidder. Since bidders will never bid B_2, the value of B_2 becomes irrelevant. From the results in subsection 4.1, it follows that $B_1 = 1$ (along with any $B_2 \geq B_1$) results in a greater expected revenue than any $B_1 > 1$ (along with any $B_2 \geq B_1$).

In order to illustrate that the expected revenue of the seller is not maximized by evenly spaced bids, it is sufficient to identify a pair of unevenly spaced bids $\left(\widetilde{B}_1, \widetilde{B}_2\right)$ which result in $\widetilde{\Pi}_S > \widehat{\Pi}^*_S$. For specified values of $\widetilde{B}_1$ and $\widetilde{B}_2$, the resulting equilibrium values of $\widetilde{c}_1$ and $\widetilde{c}_2$ can be approximated numerically. Doing so, $\widetilde{B}_1 = .65$ and $\widetilde{B}_2 = .7$ lead to $\widetilde{c}_1 \approx .33097279$ and $\widetilde{c}_2 \approx .67433844$, so that $\widetilde{\Pi}_S \approx .29624109 > .29373188 \approx \widehat{\Pi}^*_S$.

Although the expected revenue maximizing values of B_1 and B_2 have not been determined, it has been shown that the revenue maximizing bid points are not evenly spaced. Thus, the assumption in previous studies that discrete bid points must be evenly spaced is restrictive in the sense that an expected revenue maximizing seller (constrained to choosing a specific number of discrete bid points) may wish to set unevenly spaced bid points.

4.3 Some Numerical Results with Evenly Spaced Bids

Theorem 1 specifies equilibrium behavior for bidders, characterized by the unique values $c_1 < \ldots < c_k$ which simultaneously satisfy the k conditions specified by Condition (7). Because of the complexity of the conditions specifying equilibrium bidder behavior, it is not possible to solve for these values for the general case of $k + 1 > 2$ relevant discrete bid points. In order to further analyze this situation, equilibrium values of $c_1 < \ldots < c_k$ are approximated numerically, assuming bidder valuations are independently drawn from $F(v) = v^\alpha$ (with $\alpha \in (0, \infty)$).

Instead of attempting to determine the optimal (expected revenue maximizing) discrete bid points, it is simply assumed that the bids are "evenly spaced." The common distance between any two bids is denoted by t, implying $B_j = jt$.[15] As this common distance between bids is decreased, the number of acceptable bid levels below the highest possible bidder valuation may become greater. It is of particular interest to determine the behavior of both the expected revenue of the seller and the probability of ex post inefficiency as this common distance between bid points is reduced.

Approximate equilibrium values of $c_1 < \ldots < c_k$ have been determined numerically for different combinations of α and t. Of primary interest are the corresponding values of Π_S and $\Pr(I)$. When examining the value of Π_S, a comparison is made to the appropriate value of Π_T, the expected revenue from a traditional sealed bid second price auction with continuous bidding (with no reserve price). The numerical results are summarized in Tables 1 through 3 in Appendix C. It will be noticed that some of the rows in each table are "starred", i.e. they have an asterisk marking them. These rows correspond to those values of t for which $B_k = v_H = 1$.

Table 1 specifies the approximated values of Π_S, the expected revenue from the sealed bid second price auction with discrete bidding. In general Π_S is non-monotonic in t, which can be seen by considering $\alpha = \frac{1}{2}$ (or $\alpha = \frac{1}{4}$). Further, for the reported values of $\alpha \geq \frac{2}{3}$, Π_S increases as t is decreased, whereas for $\alpha = \frac{1}{10}$, Π_S decreases as t is decreased. As would be expected, Π_S appears to be monotonically increasing in α for each fixed value of t.

The results in Table 1 also allow for an examination of the change in expected revenue as a result of allowing additional bids. With $t = .5$ there are three relevant bid points: 0, $.5$, and 1. If $t = .25$, then in addition to these three bid points, bids of $.25$ and $.75$ are also acceptable. For the reported values of $\alpha \geq \frac{1}{2}$, allowing additional bids of $.25$ and $.75$ leads to an increase in the expected revenue of the seller. However, for $\alpha = \frac{1}{10}$ and $\alpha = \frac{1}{4}$, the expected revenue of the seller

[15] However, attention is not restricted to situations in which $B_k = v_H$.

decreases as a result of this change. From here it is clear that as a result of allowing additional bids, the revenue of the seller will not always increase, but may actually decrease.

The expected revenue in a traditional continuous second price auction is reported in the second to last row of Table 1. The sole purpose of calculating Π_T is to have a frame of reference to which we can compare Π_S.[16]

Table 2 reports the value of $\Pi_T - \Pi_S$ in each case considered. A negative value indicates that the seller realizes a higher expected revenue from the discrete auction under consideration, while a positive value indicates that the seller realizes a higher expected revenue from a traditional auction with continuous bidding. Since Π_T does not depend upon t, the behavior of this difference as t is decreased must mirror the behavior of Π_S as t is decreased. Additionally, while this difference typically increases as α is increased for a fixed value of t, this is not always the case (see for example the change as α is increased from $\frac{1}{10}$ to $\frac{1}{4}$ for the reported values of $t \leq \frac{1}{2}$). Finally, the negative values reported for $t \geq 1$ for $\alpha = \frac{1}{10}$ and $\alpha = \frac{1}{4}$ immediately verify the earlier observations that $\Pi_S^*(\alpha) > \Pi_T(\alpha)$ for these values of α, where Π_S^* denotes the maximum revenue of the seller when choosing the level of a single bid point $B_1 > 0$ (with $B_0 = 0$).

Finally, Table 3 reports the corresponding probability of ex post inefficiency, denoted $\Pr(I)$. This is simply the probability with which the item is awarded to the bidder with the lower valuation. $\Pr(I)$ is clearly not monotonic in α. Further, for any fixed value of α, this probability typically decreases as t becomes smaller. In fact, for the reported values this is always the case if we restrict attention to the values of t for which $B_k = v_H$ (that is, the "starred rows"). However, upon examination of Table 3 in its entirety, we see that this is not always the case. Rather, in general the probability of ex post inefficiency may actually increase as the common distance between the evenly spaced discrete bid points is decreased. For example, an increase in the value of $\Pr(I)$ is observed for both $\alpha = 4$ and $\alpha = 10$ as t is decreased from $t = 2$ to $t = 1$, from $t = 0.75$ to $t = 0.5$, from $t = 0.4$ to $t = 1/3$, and finally from $t = 0.225$ to $t = 0.2$. This provides an additional reason for concerning ourselves with situations in which $B_k \neq v_H$, since if we simply focused on cases with $B_k = v_H$ we may incorrectly infer that a decrease in the distance between evenly spaced bid points must decrease $\Pr(I)$.

5 Conclusion

A situation in which a single item is sold to two bidders by way of a sealed bid second price auction in which bids are restricted to a set of discrete values was studied. In contrast to previous studies, bids need not be "evenly spaced" and the highest acceptable bid need not be equal to the highest possible bidder valuation.

Bidder behavior in such an environment is analyzed. (Within the class of symmetric pure strategy equilibria) a unique equilibrium is shown to exist, under which bidders may choose to bid either strictly above or strictly below their independent private valuation. As a result,

[16] Recall that, as noted in footnote 12, the expected revenue from the discrete auction analyzed here must be lower than the expected revenue from the optimal auction. From Myerson (1981), a seller facing two bidders with independent private valuations drawn from $F(v) = v^\alpha$ maximizes expected revenue by imposing a reserve price of $r^* = \left(\frac{1}{1+\alpha}\right)^{1/\alpha}$ in a sealed bid second price auction. The resulting expected revenue of $\Pi_O = \frac{2\alpha}{(1+\alpha)(1+2\alpha)}\left(\alpha + \left(\frac{1}{1+\alpha}\right)^{(1+\alpha)/\alpha}\right)$ is reported in the last row of Table 1 for each α under consideration.

allocative efficiency may be sacrificed in that the item will be awarded to the bidder with the lower valuation with strictly positive probability.

Subsequently, expected seller revenue in such an auction is examined. The assumptions on the placement of the discrete bid points made in previous studies are shown to be restrictive by way of analyzing seller revenue in such an auction with only two bids and in such an auction with only three bids. With only two bid levels, an expected revenue maximizing seller may wish to set the higher acceptable bid either strictly above or strictly below the highest possible bidder valuation. Further, with only two acceptable bid levels it is possible for the expected revenue of the seller to be greater than the expected revenue in an analogous second price auction with continuous bidding (and no reserve price). Considering a situation with three discrete bid levels, when bidder valuations are uniformly distributed it is shown that having evenly spaced bids is not optimum.

Finally, a numerical analysis was conducted for an auction with an arbitrary number of evenly spaced bids. It is often the case that the seller may realize a higher expected revenue in such an auction than in a traditional auction with continuous bidding (and no reserve price). Further, the numerical analysis illustrates that the expected revenue of the seller may either increase or decrease as the common distance between bids is decreased. While the probability of ex post inefficiency often decreases as the distance between bids is decreased, this is not always the case. That is, the probability of ex post inefficiency may increase as the distance between evenly spaced discrete bids is reduced.

Appendix

A Proof of the Main Results

A.1 Proof of Proposition 1

Arbitrarily fix the behavior of the rival of bidder i. Let $\pi_i(B_m, v_i)$ denote the expected payoff for a bidder with valuation v_i from bidding B_m. It must be shown that: for a bidder with $v_i \leq B_j$, $\pi_i(B_j, v_i) \geq \pi_i(B_m, v_i)$ for every $B_m > B_j$; for a bidder with $v_i \geq B_j$, $\pi_i(B_j, v_i) \geq \pi_i(B_m, v_i)$ for every $B_m < B_j$.

Let p_j denote the probability with which the rival of bidder i submits a bid of B_j. The expected payoff of i from bidding B_m is

$$\pi_i(B_m, v_i) = \sum_{l=0}^{m-1}(v_i - B_l)p_l + (v_i - B_m)\frac{1}{2}p_m$$

Note that $\pi_i(B_m, v_i) - \pi_i(B_{m-h}, v_i)$ can be expressed as

$$\frac{1}{2}\left\{(v_i - B_m)\,p_m + (v_i - B_{m-h})\,p_{m-h}\right\} + \sum_{l=m-h+1}^{m-1}(v_i - B_l)p_l$$

for $h = 2, \ldots, m$ and as

$$\frac{1}{2}\left\{(v_i - B_m)\,p_m + (v_i - B_{m-1})\,p_{m-1}\right\}$$

for $h = 1$.

For a bidder with $v_i \leq B_{m-h}$, $\pi_i(B_m, v_i) - \pi_i(B_{m-h}, v_i) \leq 0$. Thus for a bidder with $v_i \leq B_j$, $\pi_i(B_j, v_i) \geq \pi_i(B_m, v_i)$ for all $B_m > B_j$. Similarly, for a bidder with $v_i \geq B_m$, $\pi_i(B_m, v_i) - \pi_i(B_{m-h}, v_i) \geq 0$. Thus for a bidder with $v_i \geq B_j$, $\pi_i(B_j, v_i) \geq \pi_i(B_m, v_i)$ for all $B_m < B_j$. ***Q.E.D.***

A.2 Proof of Lemma 1

It must be shown that in any symmetric, pure strategy equilibrium, $0 < p_j < 1$ for all $j = 0, 1, \ldots, k$ and $\sum_{j=0}^{k} p_j = 1$. We prove this stepwise.

Step 1. To begin with consider p_0. It can easily be seen that in any symmetric equilibrium, $0 < p_0 < 1$. Suppose, to the contrary, $p_0 = 1$. Then every bidder has an incentive to deviate unilaterally and bid B_1 and win the object for sure while paying B_0. Now suppose $p_0 = 0$. A bidder with $v_i \in [v_L, B_1)$ has a negative expected payoff in this case (since he will either obtain the item and pay more than his valuation or not obtain the item and realize a payoff of zero). By deviating to a bid of B_0 he is guaranteed a certain payoff of zero (since he will now never obtain the item). Hence it must be true that, in equilibrium, $0 < p_0 < 1$.

Step 2. Next we show that for any $j = 1, 2, \ldots, (k-1)$, if $p_0, \ldots, p_{j-1}$ are each strictly positive and $\sum_{l=0}^{j-1} p_l < 1$, then it must be true that $p_j > 0$ and $\sum_{l=0}^{j} p_l < 1$. We show this inductively.

Conjecture that there exists an equilibrium with $p_j = 0$. Focus on a bidder with $v_i \in (B_{j-1}, B_j)$. By Corollary 1 such a bidder realizes a strictly higher payoff from bidding B_{j-1} than from bidding less than B_{j-1}. Further, if $p_j = 0$ then $\pi_i(B_j, v_i) - \pi_i(B_{j-1}, v_i) = \frac{1}{2} p_{j-1}(v_i - B_{j-1})$, which is strictly positive for such a bidder. Thus, bidding B_j would result in a strictly higher payoff than bidding B_{j-1} or below.

Since $\sum_{l=0}^{j-1} p_l < 1$ it follows that positive probability must be placed on some bid above B_j. Let B_{j+m} be the lowest bid above B_j on which positive probability is placed. That is, $p_j = \ldots = p_{j+m-1} = 0$ while $p_{j+m} > 0$. It follows that $\pi_i(B_j, v_i) = \ldots = \pi_i(B_{j+m-1}, v_i) = \sum_{l=0}^{j-1}(v_i - B_l)p_l$, but $\pi_i(B_{j+m}, v_i) = \sum_{l=0}^{j-1}(v_i - B_l)p_l + (v_i - B_{j+m})\frac{1}{2}p_{j+m}$. For a bidder with $v_i \in (B_{j-1}, B_j)$ it is clear that $\pi_i(B_j, v_i) > \pi_i(B_{j+m}, v_i)$. Further, by Proposition 1 bidding B_{j+m} weakly dominates bidding B_{j+m+1} or above for such a bidder. Thus, a bidder with $v_i \in (B_{j-1}, B_j)$ would strictly increase his expected payoff by deviating from the conjectured equilibrium and instead submitting a bid of B_j. Hence, in equilibrium we cannot have $p_j = 0$, but must rather have $p_j > 0$.

Now conjecture that $\sum_{l=0}^{j} p_l = 1$, which would imply that all bids above B_j are submitted with zero probability. From here, bidding B_j leads to a payoff of $\pi_i(B_j, v_i) = \sum_{l=0}^{j-1}(v_i - B_l)p_l + (v_i - B_j)\frac{1}{2}p_j$ while bidding B_{j+1} or above leads to $\pi_i(B_{j+1}, v_i) = \ldots = \pi_i(B_k, v_i) =$

$\sum_{l=0}^{j}(v_i - B_l)p_l$. The latter of these is clearly larger for any bidder with $v_i \geq B_j$. Further, by Proposition 1 bidding B_j weakly dominates bidding B_{j-1} or below for such a bidder. Thus, a bidder with $v_i \geq B_j$ could strictly increase his expected payoff by deviating from the conjectured equilibrium and instead submitting a bid above B_j. Hence, it must be true that $\sum_{l=0}^{j} p_l < 1$ in equilibrium.

Given that $p_0 > 0$, the above has to hold for $j = 1$. Similarly, it has to hold for all $j = 1, 2, \ldots, (k-1)$.

Step 3. Finally, if $\sum_{l=0}^{k-1} p_l < 1$, then it must be true that $p_k > 0$ and $\sum_{l=0}^{k} p_l = 1$. To see why $p_k > 0$, note that if $p_k = 0$ then a bidder with valuation $v_i = v_H$ would have a higher payoff from bidding B_k as opposed to anything less (since $\pi_i(B_k, v_H) = \sum_{l=0}^{k-1}(v_H - B_l)p_l$ is greater than $\pi_{B_{k-1}, v_H} = \sum_{l=0}^{k-2}(v_H - B_l)p_l + (v_i - B_{k-1})\frac{1}{2}p_{k-1}$, along with the fact that bidding B_{k-1} weakly dominates bidding below B_{k-1} for such a bidder). To see why $\sum_{l=0}^{k} p_l = 1$, first note that either B_k is the highest acceptable bid or there are other acceptable bids above B_k. In the former case all weight must be on bids B_0 through B_k, since there are no other acceptable bids. In contrast, if there are other acceptable bids above B_k, then (by the way in which B_k was defined) it must be that $B_k \geq v_H$. By Corollary 2 it follows that in this case bidding B_{k+1} or above results in a strictly lower payoff than bidding B_k.

This proves our claim that in any symmetric, pure strategy equilibrium, $0 < p_j < 1$ for all $j = 0, 1, \ldots, k$ and $\sum_{j=0}^{k} p_j = 1$. *Q.E.D.*

A.3 Proof of Theorem 1

It must be shown that, for arbitrary acceptable bid levels, there exist unique values C such that condition (7) holds for $j = 1, \ldots, k$; and for this unique C, neither bidder has an incentive to unilaterally deviate from the conjectured strategy pair.

For $D_k(c_{k-1}, c_k, c_{k+1}) = (B_k - c_k)[1 - F(c_k)] - (c_k - B_{k-1})[F(c_k) - F(c_{k-1})]$ observe that $D_k(c_{k-1}, min\{v_H, B_k\}, c_{k+1}) < 0$ and $D_k(c_{k-1}, B_{k-1}, c_{k+1}) > 0$. Further, $\frac{\partial D_k}{\partial c_k} = -[1 - F(c_{k-1})] - f(c_k)(B_k - B_{k-1}) < 0$, implying that for any $c_{k-1} < B_{k-1}$ there exists a unique $c_k \in (B_{k-1}, min\{v_H, B_k\})$ such that $D_k(c_{k-1}, c_k, c_{k+1}) = 0$. Letting $c_k(c_{k-1})$ denote this value, by the Implicit Function Theorem

$$c_k'(c_{k-1}) = \frac{(c_k - B_{k-1})f(c_{k-1})}{[1 - F(c_{k-1})] + (B_k - B_{k-1})f(c_k)}.$$

Clearly $c_k'(c_{k-1}) \in \left(0, \frac{f(c_{k-1})}{f(c_k(c_{k-1}))}\right)$.

Now consider

$$D_j(c_{j-1}, c_j, c_{j+1}(c_j)) = (B_j - c_j)[F(c_{j+1}(c_j)) - F(c_j)] - (c_j - B_{j-1})[F(c_j) - F(c_{j-1})],$$

presuming that an examination of $D_{j+1}(c_j, c_{j+1}, c_{j+2}(c_{j+1}))$ has lead to the defining of $c_{j+1}(c_j) \in (B_j, B_{j+1})$ such that $c'_{j+1}(c_j) \in \left(0, \frac{f(c_j)}{f(c_{j+1}(c_j))}\right)$. Note that: $D_j(c_{j-1}, B_{j-1}, c_{j+1}(B_{j-1})) > 0$ and $D_j(c_{j-1}, B_j, c_{j+1}(B_j)) < 0$. Further, (since $c'_{j+1}(c_j) < \frac{f(c_j)}{f(c_{j+1}(c_j))}$):

$$\begin{aligned}\frac{\partial D_j}{\partial c_j} &= -[F(c_{j+1}(c_j)) - F(c_{j-1})] - f(c_j)(B_j - B_{j-1}) + (B_j - c_j) f(c_{j+1}(c_j))\, c'_{j+1}(c_j) \\ &< -[F(c_{j+1}(c_j)) - F(c_{j-1})] - f(c_j)(c_j - B_{j-1}) < 0.\end{aligned}$$

Thus, for any $c_{j-1} < B_{j-1}$ there exists a unique $c_j \in (B_{j-1}, B_j)$ for which $D_j(c_{j-1}, c_j, c_{j+1}(c_j)) = 0$. Letting $c_j(c_{j-1})$ denote this value,

$$c'_j(c_{j-1}) = \frac{(c_j - B_{j-1}) f(c_{j-1})}{[F(c_{j+1}(c_j)) - F(c_{j-1})] + f(c_j)(B_j - B_{j-1}) - (B_j - c_j) f(c_{j+1}(c_j))\, c'_{j+1}(c_j)}.$$

Since $c'_{j+1}(c_j) \in \left(0, \frac{f(c_j)}{f(c_{j+1}(c_j))}\right)$, it follows that $c'_j(c_{j-1}) \in \left(0, \frac{f(c_{j-1})}{f(c_j(c_{j-1}))}\right)$.

Finally, consider $D_1(c_0, c_1, c_2(c_1))$ (supposing that an examination of $D_2(c_1, c_2, c_3(c_2))$ has led to the defining of $c_2(c_1)$ such that $c'_2(c_1) \in \left(0, \frac{f(c_1)}{f(c_2(c_1))}\right)$). First note that $D_1(c_0, c_0, c_2(c_0)) > 0$ and $D_1(c_0, B_1, c_2(B_1)) < 0$. Further,

$$\begin{aligned}\frac{\partial D_1}{\partial c_1} &= -F(c_2(c_1)) - f(c_1)(B_1 - B_0) + (B_1 - c_1) f(c_2(c_1))\, c'_2(c_1) \\ &< -F(c_2(c_1)) - f(c_1)(c_1 - B_0) < 0.\end{aligned}$$

Thus, there exists a unique $c_1 \in (c_0, B_1)$ for which $D_1(c_0, c_1, c_2(c_1)) = 0$.

From here it follows that, for any arbitrary bid levels, there exists a unique vector $C = (c_0, c_1, \ldots, c_k, c_{k+1})$, with $c_0 = v_L$, $c_j \in (B_{j-1}, B_j)$ for $j = 1, \ldots, k-1$, $c_k \in (B_{k-1}, min\{v_H, B_k\})$, and $c_{k+1} = v_H$, such that condition (7) holds for $j = 1, \ldots, k$ simultaneously.

It is clear that for this unique C, neither bidder wishes to unilaterally deviate from the conjectured strategy pair. *Q.E.D.*

A.4 Proof of Proposition 2

Defining $g(\alpha) = (1+\alpha)^{\frac{\alpha-1}{\alpha}}$, we have $B_1^*(\alpha) = \frac{g(\alpha)}{\alpha}$. It follows that $B_1^*(1) = 1$.

Since

$$g'(\alpha) = (1+\alpha)^{-\frac{1}{\alpha}} \left\{ \frac{(1+\alpha)\ln(1+\alpha) + \alpha(\alpha-1)}{\alpha^2} \right\},$$

we have that

$$\frac{dB_1^*(\alpha)}{d\alpha} = \frac{\alpha g'(\alpha) - g(\alpha)}{\alpha^2} = \frac{(1+\alpha)\ln(1+\alpha) - 2\alpha}{(1+\alpha)^{\frac{1}{\alpha}}\, \alpha^3}.$$

The sign of this expression is determined by the sign of

$$\eta(\alpha) = (1+\alpha)\ln(1+\alpha) - 2\alpha.$$

Note that $\eta(0) = 0$. Further, $\eta'(\alpha) = \ln(1+\alpha) - 1$, which implies $\eta'(0) = -1 < 0$. Additionally, $\eta''(\alpha) = \frac{1}{1+\alpha} > 0$. From here we have that there exists a unique $\tilde{\alpha} > 0$ such that $\eta(\alpha) < 0$ for

$\alpha \in (0, \tilde{\alpha})$ and $\eta(\alpha) > 0$ for $\alpha \in (\tilde{\alpha}, \infty)$. As a result, $B_1^*(\alpha)$ is decreasing in α for $\alpha \in (0, \tilde{\alpha})$ and increasing in α for $\alpha \in (\tilde{\alpha}, \infty)$. That is, $B_1^*(\alpha)$ is a quasi-convex function.

Since $B_1^*(\alpha)$ is less than $\frac{1+\alpha}{\alpha}$, and $\lim_{\alpha \to \infty} \frac{1+\alpha}{\alpha} = 1$, it follows that $\lim_{\alpha \to \infty} B_1^*(\alpha) \leq 1$. From here, by the quasi-convexity of $B_1^*(\alpha)$, $B_1^*(\alpha) > 1 = v_H$ for $\alpha < 1$ and $B_1^*(\alpha) < 1 = v_H$ for $\alpha > 1$. *Q.E.D.*

A.5 Proof of Proposition 3

Consider the function, $B_1^E(\alpha) = 2^{\frac{\alpha-1}{\alpha}}$. Begin by noting that $B_1^E(1) = 1$. Further,

$$\frac{dB_1^E(\alpha)}{d\alpha} = \frac{2^{\frac{\alpha-1}{\alpha}} \ln(2)}{\alpha^2},$$

which is strictly positive for all $\alpha \in (0, \infty)$. From here: $B_1^E(\alpha) < 1$ for $\alpha < 1$, and $B_1^E(\alpha) > 1$ for $\alpha > 1$. *Q.E.D.*

B Figures

Figure 1: Optimal choice of B_j vs. B_{j-1}.

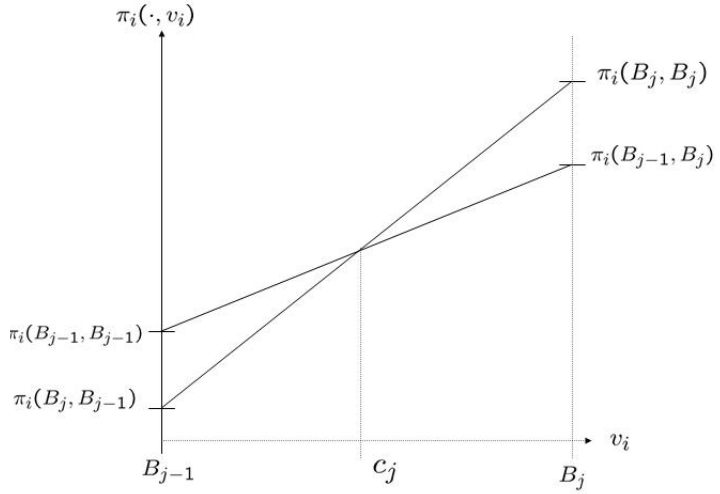
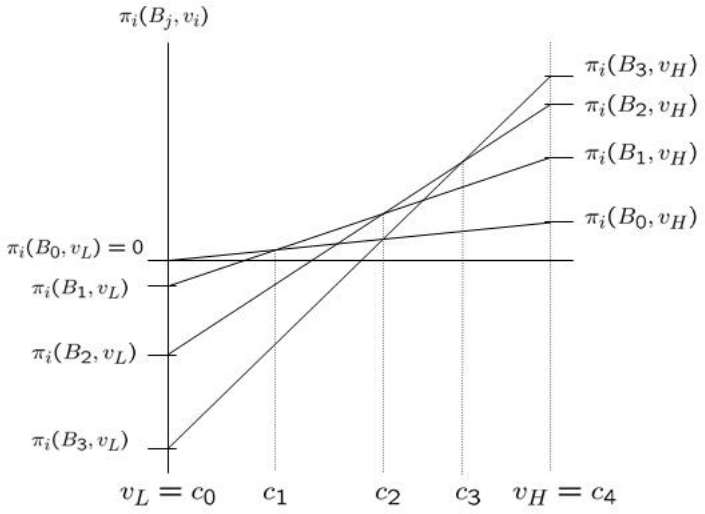


Figure 2: Bidder Equilibrium with four bid points.



C Numerical Results

Table 1: Seller's Revenue for Discrete Auction

	Alpha Values								
Increment	1/10	1/4	1/2	2/3	1	3/2	2	4	10
2.00	0.032580	0.081913	0.143594	0.174756	0.222222	0.271088	0.304806	0.376253	0.440493
1.00*	0.027199	0.075905	0.145898	0.185037	0.250000	0.324718	0.381966	0.524889	0.697357
0.75	0.026213	0.077005	0.153974	0.198259	0.272940	0.359697	0.426100	0.587733	0.758068
0.50*	0.024416	0.076355	0.159726	0.208988	0.292893	0.390059	0.463021	0.629304	0.776064
0.4	0.023502	0.076079	0.163088	0.215176	0.304222	0.406902	0.483094	0.652247	0.803395
1/3*	0.022719	0.075460	0.164526	0.218219	0.310000	0.415138	0.492292	0.660467	0.808426
0.30	0.022293	0.075155	0.165485	0.220181	0.313749	0.420712	0.498903	0.668410	0.817459
0.25*	0.021560	0.074468	0.166496	0.222539	0.318386	0.427401	0.506495	0.676162	0.823800
0.225	0.021158	0.074076	0.167026	0.223812	0.320955	0.431266	0.511122	0.682126	0.830921
0.2*	0.020717	0.073589	0.167361	0.224805	0.323026	0.434252	0.514487	0.685489	0.833130
0.15	0.019728	0.072432	0.167890	0.226697	0.327186	0.440521	0.521947	0.695119	0.844307
0.125*	0.019164	0.071718	0.167960	0.227391	0.328865	0.443033	0.524885	0.698661	0.847488
0.1*	0.018546	0.070913	0.167927	0.227966	0.330379	0.445357	0.527681	0.702526	0.852292
0.0825	0.018074	0.070287	0.167827	0.228269	0.331279	0.446754	0.529376	0.704981	0.855666
1/15*	0.017616	0.069674	0.167680	0.228467	0.331964	0.447822	0.530673	0.706899	0.858417
0.06	0.017412	0.069402	0.167604	0.228531	0.332220	0.448227	0.531172	0.707693	0.859798
0.05*	0.017094	0.068977	0.167470	0.228597	0.332547	0.448741	0.531795	0.708624	0.861163
0.04*	0.016759	0.068534	0.167319	0.228637	0.332824	0.449181	0.532332	0.709473	0.862624
0.025*	0.016217	0.067836	0.167064	0.228646	0.333131	0.449671	0.532933	0.710444	0.864424
Traditional	0.015152	0.066667	0.166667	0.228571	0.333333	0.450000	0.533333	0.711111	0.865801
Optimal	0.068257	0.154048	0.265432	0.324179	0.416667	0.515146	0.584653	0.734889	0.871994

Table 2: Revenue Loss from Discrete Auction

	Alpha Values								
Increment	1/10	1/4	1/2	2/3	1	3/2	2	4	10
2	-0.017429	-0.015246	0.023073	0.053815	0.111111	0.178912	0.228527	0.334858	0.425308
1.00*	-0.012047	-0.009238	0.020769	0.043534	0.083333	0.125282	0.151367	0.186223	0.168444
0.75	-0.011061	-0.010339	0.012692	0.030312	0.060394	0.090303	0.107233	0.123379	0.107733
0.50*	-0.009265	-0.009688	0.006940	0.019584	0.040440	0.059941	0.070313	0.081807	0.089737
0.4	-0.008350	-0.009413	0.003578	0.013395	0.029112	0.043098	0.050239	0.058864	0.062406
1/3*	-0.007567	-0.008794	0.002140	0.010353	0.023333	0.034862	0.041042	0.050644	0.057375
0.3	-0.007141	-0.008488	0.001182	0.008390	0.019585	0.029288	0.034431	0.042701	0.048342
0.25*	-0.006409	-0.007801	0.000171	0.006032	0.014947	0.022599	0.026838	0.034949	0.042001
0.225	-0.006006	-0.007409	-0.000359	0.004759	0.012378	0.018734	0.022211	0.028985	0.034880
0.20*	-0.005566	-0.006922	-0.000694	0.003767	0.010307	0.015748	0.018846	0.025622	0.032671
0.15	-0.004577	-0.005765	-0.001224	0.001874	0.006148	0.009479	0.011386	0.015992	0.021494
0.125*	-0.004013	-0.005051	-0.001293	0.001180	0.004469	0.006967	0.008448	0.012450	0.018313
0.1*	-0.003394	-0.004246	-0.001260	0.000605	0.002955	0.004643	0.005652	0.008585	0.013509
0.0825	-0.002923	-0.003621	-0.001160	0.000302	0.002054	0.003246	0.003957	0.006130	0.010135
1/15*	-0.002464	-0.003007	-0.001013	0.000105	0.001369	0.002178	0.002660	0.004212	0.007384
0.06	-0.002261	-0.002735	-0.000937	0.000040	0.001113	0.001773	0.002162	0.003419	0.006003
0.05*	-0.001943	-0.002311	-0.000803	-0.000026	0.000786	0.001259	0.001539	0.002487	0.004638
0.04*	-0.001608	-0.001868	-0.000652	-0.000066	0.000509	0.000819	0.001001	0.001638	0.003177
0.025*	-0.001066	-0.001169	-0.000397	-0.000075	0.000202	0.000329	0.000401	0.000667	0.001377

Table 3: Probability of ex-post Inefficiency

	Alpha Values								
Increment	1/10	1/4	1/2	2/3	1	3/2	2	4	10
2	0.388657	0.338580	0.303848	0.291780	0.277778	0.267381	0.262015	0.254391	0.250942
1.00*	0.362278	0.300397	0.263932	0.254878	0.250000	0.254878	0.263932	0.300397	0.362278
0.75	0.345442	0.271880	0.226490	0.214226	0.205377	0.206527	0.212638	0.238715	0.274686
0.50*	0.322774	0.236742	0.187015	0.175738	0.171573	0.180910	0.194430	0.241304	0.309451
0.4	0.309726	0.216003	0.161817	0.149174	0.142928	0.148877	0.158548	0.190508	0.231779
1/3*	0.299417	0.200690	0.145579	0.133785	0.130000	0.139889	0.153454	0.200522	0.275591
0.3	0.293415	0.191584	0.134985	0.122703	0.117867	0.125637	0.136732	0.173879	0.227501
0.25*	0.283315	0.177007	0.119625	0.107819	0.104135	0.113494	0.126241	0.171559	0.248760
0.225	0.277550	0.168723	0.110524	0.098472	0.093990	0.101466	0.111972	0.147954	0.202218
0.20*	0.271265	0.160085	0.101899	0.090284	0.086612	0.095177	0.106935	0.149843	0.226908
0.15	0.256371	0.140215	0.082025	0.070757	0.066710	0.073190	0.082406	0.116084	0.174288
0.125*	0.247336	0.128858	0.071535	0.060859	0.057267	0.063687	0.072831	0.108325	0.180142
0.1*	0.236672	0.116042	0.060139	0.050113	0.046651	0.052062	0.059927	0.091306	0.158541
0.0825	0.227830	0.105944	0.051650	0.042239	0.038906	0.043448	0.050172	0.077401	0.137272
1/15*	0.218406	0.095728	0.043575	0.034916	0.031845	0.035679	0.041446	0.065434	0.122034
0.06	0.213885	0.091018	0.039994	0.031687	0.028674	0.032020	0.037120	0.058177	0.106225
0.05*	0.206278	0.083402	0.034517	0.026882	0.024163	0.027115	0.031648	0.050935	0.099178
0.04*	0.197325	0.074899	0.028747	0.021896	0.019464	0.021860	0.025589	0.041680	0.083516
0.025*	0.179694	0.059634	0.019452	0.014135	0.012291	0.013817	0.016246	0.026959	0.056635

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