A Normalized Spence-Dixit-Stiglitz Index

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Abstract

This note aims at providing a different view on the well known Spence-Dixit-Stiglitz index. This is helpful when interpreting this index in the form usually used in economic models. A normalization procedure based on limiting properties of marginal products is proposed and it is recommended to use it when drawing inference about the elasticity of substitution.

Keywords: Dixit-Stiglitz Index, Normalized CES function, Elasticity of Substitution.

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1 Introduction

The Spence-Dixit-Stiglitz (SDS) index has since its introduction by Spence (1976) and Dixit and Stiglitz (1977) proven to be a powerful building stone in economic models. In its original application it was used to analyze competition related questions. However, soon Krugman (1979, 1980) employed the approach to found the new trade theory and later the new economic geography (Krugman 1990). Grossman and Helpman (1991) introduced it into endogenous growth theory while Young (1998) used it in a semi-endogenous growth model. Ethier (1982) generalized the SDS index to elaborate more on the effects of product differentiation. Today, the SDS index can be found in all areas of modern economics. The index is used to model consumer preferences as well as to specify production technologies.

The SDS index has the well known property of exhibiting a constant elasticity of substitution between its input factors. As such it is strongly related to the CES production function formally established in Arrow et al. (1961). However, the CES production function has explicitly derived from a certain underlying theory on the relationship between the marginal product of labor and production using capital and labor as factors. Such an underlying derivation is not available for the SDS index which was introduced more ad hoc.

Looking closer at recent theoretical research on the CES production function might be helpful in interpreting the ad hoc structure of the SDS index. The normalized CES production function (de La Grandville 1989) has recently gained increasing attention. It is extensively discussed in de La Grandville (2009). Klump and Saam (2008) used it for drawing conclusions about the influence of the elasticity of substitution on the convergence speed in the Ramsey model. Papageorgiou and Saam (2008) employed it in a nested CES production framework for high and low skilled labor. Jones (2003) and Antony (2009a,b, 2010) used it for constructing "functional" normalized production functions. Klump and de La Grandville (2000) and Klump and Preissler (2000) advocate its general use in growth theory. For a critical discussion of the normalization procedure see Temple (2008).

This critical view is taken up in this note to establish an alternative way of normalizing using limiting properties of marginal products. This procedure has the advantage of having a more direct economic interpretation and it turns out that it is helpful in interpreting the well known SDS index. In addition a general normalized SDS index is proposed and its implications and advantages will be discussed.

The next section deals with the normalization of CES production functions and establishes the alternative normalization procedure. Section 3 applies this normalization to the SDS index and Section 4 provides a practical application. Finally, the last section concludes.

2 The Theory of Normalization

The normalized CES production function for two input factors, which we will call labor L and capital K for concreteness, can be written as

$$y = \bar{y}\left(\frac{\bar{k}^{1-\rho}+\bar{\mu}}{\bar{k}+\bar{\mu}}\right)^{\frac{1}{\rho}}\left(\frac{\bar{k}^{1-\rho}}{\bar{k}^{1-\rho}+\bar{\mu}}k^{1-\rho}+\frac{\bar{\mu}}{\bar{k}^{1-\rho}+\bar{\mu}}\right)^{\frac{1}{\rho}}, \tag{1}$$

or equivalently as

$$y = \bar{y}\left(\bar{\pi}\left(\frac{k}{\bar{k}}\right)^{\rho}+(1-\bar{\pi})\right)^{\frac{1}{\rho}}, \tag{2}$$

where y is output per worker, $k=\frac{K}{L}$ is the capital intensity and $\sigma=\frac{1}{1-\rho}$ is the elasticity of substitution. The normalization involves either choosing $\bar{y}$, $\bar{k}$ and $\bar{\mu}$ or $\bar{\pi}$, i.e. the baseline level of production per worker, the baseline capital intensity, the baseline marginal rate of substitution between labor and capital and the baseline capital share in production.

The baseline values enter the analysis when determining the constants of integration in the CES production function obtained by Arrow et al. (1961) from baseline conditions as is explained in detail in e.g. de La Grandville (2009). Yet, this method does not guide us how to chose these values practically.

Another form of normalization can be undertaken by observing the limiting properties of the general CES function as obtained in Arrow et al. (1961)

$$y = A(\rho)\left[\alpha(\rho)k^{\rho}+(1-\alpha(\rho))\right]^{\frac{1}{\rho}}. \tag{3}$$

For $\rho < 0$, i.e. $\sigma < 1$, the marginal product of capital for $k \to 0$ is given by

$$\lim_{k \to 0} \frac{\partial y}{\partial k} = A(\rho)\alpha(\rho)^{\frac{1}{\rho}},$$

for $\rho > 0$, i.e. $\sigma > 1$, the same expression gives the marginal product of capital for $k \to \infty$.

Analogous calculations for the other factor labor reveal that its marginal product approaches $A(\rho)(1-\alpha(\rho))^{\frac{1}{\rho}}$ for $\rho < 0$ as $k \to \infty$ and for $\rho > 0$ as $k \to 0$.

Denoting the limiting marginal products for capital and labor as $\bar{r}$ and $\bar{w}$, the CES function (3) in normalized form becomes

$$y = \left[(\bar{r}k)^{\rho} + \bar{w}^{\rho}\right]^{\frac{1}{\rho}}.$$

The advantage of normalizing the CES production function this way is that the normalized form has a clear technological interpretation as it uses the limits of marginal productivities. In the normalizations (1) and (2) it is not completely clear what the economic meaning of the baseline values are. It is not obvious where they are realized and what particular value $\bar{k}$ takes[1].

3 Normalizing the SDS Index

The SDS index, denoted by Y_{SDS}, is given by

$$Y_{SDS} = \left(\int_0^N x_i^{\rho} di\right)^{\frac{1}{\rho}}, \tag{4}$$

where Y_{SDS} might serve as a (sub-) production or utility function in an economic model. N is the mass of differentiated input factors which input quantity is denotes by x_i.

[1] Jones (2003) terms $\bar{k}$ an "appropriate" value for the capital intensity, Klump et al. (2007) think of at it in terms of a long run average, Klump and Saam (2006) interpret it as a steady state and Temple (2008) criticizes that there seems no clear concept of how to interpret it.

For normalizing the SDS index it is useful at this point to introduce a "generalization" of the SDS index

$$\tilde{Y}_{SDS} = \tilde{A}(\rho) \left[\int_0^N \alpha_i(\rho) x_i^\rho di \right]^{\frac{1}{\rho}}, \tag{5}$$

where $\tilde{A}(\rho) > 0$ and $\int_0^N \alpha_i(\rho) di = 1$.

The marginal product χ_j of one particular variant $j \in (0, N)$ of the input factors in (5) is

$$\chi_j = \tilde{A}(\rho) \left(\int_0^N \alpha_i(\rho) x_i^\rho di \right)^{\frac{1-\rho}{\rho}} \alpha_j(\rho) x_j^{\rho-1}.$$

It is easy to verify that for $\rho < 0$ and $x_j \to 0$ the marginal product approaches

$$\lim_{x_j \to 0} \chi_j = \tilde{A}(\rho) \alpha_j(\rho)^{\frac{1}{\rho}} = \bar{\chi}_j.$$

The same expression gives the marginal product of variant j for $\rho > 0$ and $x_j \to \infty$. Using this result in (5) gives

$$\tilde{Y}_{SDS} = \left[\int_0^N (\bar{\chi}_j x_i)^\rho di \right]^{\frac{1}{\rho}}. \tag{6}$$

Comparing (6) with (4) reveals that the well known SDS index is an implicitly normalized CES index with limiting marginal products equal to one. As $\rho > 0$ is usually assumed in models using the SDS index, a researcher implicitly assumes by doing so that the marginal product of all input factors approaches unity as the input quantity goes to infinity.

The reason for this result being important is the same reason that led de La Grandville and Klump (2000) and Klump and Preissler (2000) to conclude that the normalized CES function, either (1) or (2), should be used in general. If a researcher wants to draw inference about the influence of the elasticity of substitution σ on her results this can in general only be done using the normalization approach. The only exception for the CES production function is the case $\bar{k} = 1$ when σ, and hence ρ, has no influence on $A(\rho)$ and $\alpha(\rho)$. If $\bar{k} \neq 1$, a change in the elasticity of substitution has a deeper influence on production. The same argument holds for the SDS index. Only if it happens to be the case that $\bar{\chi}_i = 1$ for all i a valid analysis of the influence of σ on the results can be undertaken. In all other cases the inference might be misleading.

This argument can best be seen by formulating (6) as

$$\tilde{Y}_{SDS} = \left(\int_0^N \bar{\chi}_i^{\rho} di\right)^{\frac{1}{\rho}} \left(\int_0^N \frac{\bar{\chi}_i^{\rho}}{\int_0^N \bar{\chi}_j^{\rho} dj} x_i^{\rho} di\right)^{\frac{1}{\rho}}, \tag{7}$$

which reveals

$$\begin{aligned} \tilde{A}(\rho) &= \left(\int_0^N \bar{\chi}_i^{\rho} di\right)^{\frac{1}{\rho}}, \\ \alpha_i(\rho) &= \frac{\bar{\chi}_i^{\rho}}{\int_0^N \bar{\chi}_j^{\rho} dj}. \end{aligned}$$

We clearly see the influence of the elasticity of substitution on the structure of the SDS index if $\bar{\chi}_i \neq 1$.

4 An Application

As an illustrative application of the above normalized SDS index, the Armington (1969) aggregator is chosen which is frequently used in trade models (see e.g. Balistreri et al. (2010) for a recent example). The simple Armington aggregator used in Boileau (1999) and Backus et al. (1994) might serve here as a good starting point to get an impression about the impact of the proposed normalization.

In Boileau (1999), the world economy consists of two countries, 1 and 2, both are engaged in producing final output which is traded among them. The total volume of final output that is available in country 1 is defined as[2]

$$G(Y_1, Y_2^*) = [\omega(Y_1)^a + (1-\omega)(Y_2^*)^a]^{\frac{1}{a}}, \tag{8}$$

[2]In original notation from Boileau (1999).

where Y_1 is production of final output in country 1 which is not exported and Y_2^* is final output produced in country 2 and exported to country 1.
Mathematically, (8) is a discrete version of (6) which can be rewritten as

$$G(Y_1, Y_2^*) = \left[(\omega^{\frac{1}{a}} Y_1)^a + ((1-\omega)^{\frac{1}{a}} Y_2^*)^a\right]^{\frac{1}{a}}, \tag{9}$$

with $\chi_1 = \omega^{\frac{1}{a}}$, $\chi_2 = (1-\omega)^{\frac{1}{a}}$ and $\rho = a$. Boileau (1999) and Backus et al. (1994) use $\omega = 0.8$ and $\frac{1}{1-a} = 1.5$ for their model calibration. In the spirit of the above normalization $\chi_1 = \omega^{\frac{1}{a}}$ and $\chi_2 = (1-\omega)^{\frac{1}{a}}$ are the limiting marginal products which are constants. Thus, if e.g. a sensitivity analysis with respect to a is done, ω and $(1-\omega)$ have to be adjusted in order to leave χ_1 and χ_2 unchanged. Not taking account of the above normalization idea, would lead the researcher to chose ω constant and to change χ_1 and χ_2. In order to better compare these two competing ideas, we rewrite (9) as

$$\frac{G(Y_1, Y_2^*)}{Y_2^*} = \left[(\omega^{\frac{1}{a}} Y_1/Y_2^*)^a + (1-\omega)^{\frac{1}{a}}\right]^{\frac{1}{a}}, \tag{10}$$

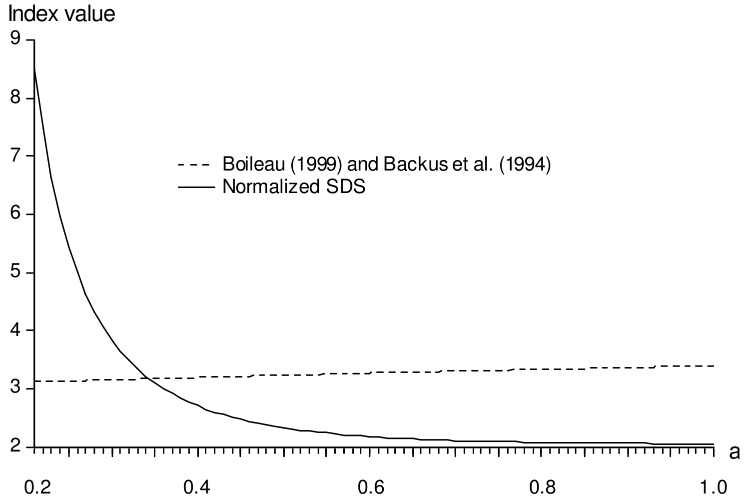
Given a relative openness to trade for the U.S. of about 0.25 during 2000 to 2007[3], a value of 4 is chosen for $\frac{Y_1}{Y_2^*}$. Figure 1 below shows the behavior of (10) around $\frac{1}{1-a} = 1.5$ holding either ω or χ_1 and χ_2 constant.
The difference between the two calibration exercises is striking. While in the formalization of Boileau (1999) and Backus et al. (1994), the index given by (10) is increasing in a, it is decreasing if we use the proposed normalization procedure. Of course at $\frac{1}{1-a} = 1.5$ the two curves intersect. Note again that doing the normalization, this has a solid foundation in a proper economic interpretation in terms of marginal products.

5 Discussion and Conclusion

This note presented, first, an alternative way of normalizing a CES function and, second, used it to elaborate on the implicit assumptions of the well known and

[3]See e.g. the Penn World Tables 6.3 available at http://pwt.econ.upenn.edu/.

Figure 1: Sensitivity analysis



Note: Sensitivity analysis of (10) with respect to $a \in (0.2, 1.0)$. $a = 1/3$ corresponds to the case considered in Boileau (1999) and Backus et al. (1994). "Boileau (1999) and Backus et al. (1994)" holds ω constant, while the case "Normalized SDS" holds χ_1 and χ_2 constant.

frequently used SDS index. The ad hoc specification of this index normally used implies strong assumptions about the limiting properties of marginal products of input factors.

If the believes of a researcher are not in accordance with these assumptions, the use of the normalized SDS is advocated. This is especially in order when drawing inference about the degree of substitutability between input factors. As this elasticity of substitution is commonly interpreted as being related to the degree of market competitiveness, this is frequently done. A good example is the Armington (1969) aggregator which is just a special case of (7) and is heavily used in CGE models as recently e.g. in Balistreri et al. (2010). The application above has shown the severe consequences of normalization.

Also some words on the use of the SDS index in growth theory are in order. Especially in quality ladder models as in Young (1998) one reads specifications as $Y = (\int_0^N (\lambda_{i,t} x_{i,t})^\rho di)^{1/\rho}$, where $\lambda_{i,t}$ is the quality of an input factor at time t and $x_{i,t}$ is the quantity employed. Using the normalized SDS index we readily can interpreter $\lambda_{i,t}$ as the limiting marginal product if $\rho > 0$. Growth through increasing quality is therefore identical to technological changes that alter the limiting marginal products of input factors.

As the above example of quality ladders points out, the normalized SDS index is helpful in interpreting assumptions. But more important, it is necessary when drawing conclusions about the influence of the elasticity of substitution on results.

Literature

Antony, J., 2009a, A Dual Elasticity of Substitution Production Function with an Application to Cross Country Inequality. Economics Letters, 102, 10-12.

Antony, J., 2009b, Capital/Labor Substitution, Capital Deepening, and FDI. Journal of Macroeconomics, 31, 699-707.

Antony, J., 2010, A Class of Changing Elasticity of Substitution Production Functions. Journal of Economics, 100, 165-183.

Armington, P. S., 1969, A Theory of Demand for Products Distinguished by Place of Production, Staff Papers - International Monetary Fund, 16, 159-178.

Arrow, K. J., H. B. Chenery, B. S. Minhas and R. M. Solow, 1961, Capital-Labor Substitution and Economic Efficiency, Review of Economics and Statistics 43, 225-250.

Backus, D., P. Kehoe and F. Kydland, 1994, Dynamics of the Trade Balance and the Terms of Trade: The J-Curve, American Economic Review 84, 84-103.

Balistreri, E. J., R. H. Hillberry and T. F. Rutherford, 2010, Trade and Welfare: Does Industrial Organization Matter? Economics Letters, 109, 85-87.

Boileau, M., 1999, Trade in Capital Goods and the Volatility of Net Exports and the Terms of Trade, Journal of International Economics, 48, 347-365.

de La Grandville, O., 2009, Economic Growth: A Unified Approach, Cambridge University Press.

de La Grandville, O., 1989, In the Quest for the Slutzky Diamond, American Economic Review 79, 468-481.

Jones, C. I., 2003, Growth, Capital Shares, and a New Perspective on Production Functions, University of California, Berkeley.

Klump, R. and O. de La Grandville, 2000, Economic Growth and the Elasticity of Substitution: Two Theorems and Some Suggestions, American Economic Review 90, 282-291.

Klump, R., P. McAdam and A. Willman, 2007, The long-term SucCESs of the Neoclassical Growth Model, Oxford Review of Economic Policy, 23, 94-114.

Klump, R. and H. Preissler, 2000, CES Production Functions and Economic Growth, Scandinavian Journal of Economics 102, 41-56.

Klump, R. and M. Saam, 2008, Calibration of Normalised CES Production Functions in Dynamic Models, Economics Letters, 99, 256-259

Papageorgiou, C. and M. Saam, 2008, Two-Level CES Production Technology in the Solow and Diamond Growth Models, Scandinavian Journal of Economics, 110, 119-143.

Temple, J. R. W., 2008, The Calibration of CES Production Functions. Department of Economics, University of Bristol, Working Paper no. 08/606.