Yield to Maturity and Total Rate of Return: A Theoretical Note*

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Abstract

This note provides a formal analysis of the connection between the yield to maturity (YTM) and the total rate of return of a coupon bond. It shows that the YTM precisely measures the (annual) total rate of return with the bond valuated by its *amortized book value* - the present value of the remaining cash inflows discounted at the initial YTM.

***Key Words*:** Yield to maturity; (Total) rate of return; Amortized book value
JEL codes: E40, G00

1. Introduction

"Yield to maturity" (hereafter YTM) and "total rate of return" both measure the interest rate on a debt instrument but are differently defined in mathematics. On one hand, YTM is defined as the discount rate that "equates the present value of payments received from a debt instrument with its value today" (e.g., Mishkin, 2004, p. 64). By its very nature, for a coupon bond the YTM is *internally* determined by its face value, the coupon payments, the current term to maturity, and the initial purchase price, regardless how the market price of the bond actually evolves during the holding period. Hence, YTM serves as a yardstick to compare the returns from different types of debt instruments (e.g., Hubbard, 2004, p. 65; Ross, Westerfield, and Jordan, 2008, p. 191). On the other hand, the (annual) "total rate of return" from holding a bond is defined as the current yield plus the rate of capital gain, depending on the market price of the bond between two consecutive years. Although they are defined differently in mathematics, YTM is "widely viewed as a measure of the average rate of return from holding a bond until maturity" (Bodie *et al*, 2002, p. 426). In fact, some authors even define YTM as "the rate of return if the bond is held until maturity" (e.g., Miller and VanHoose, 2004, p.82). To our knowledge, only Campbell e*t al* (1997, pp. 397-399) have formally shown that for zero-coupon bond the (gross) YTM equals to the geometric average of (gross) rate of

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* We would like to thank two anonymous referees and an Associate Editor for comments and suggestions. The standard disclaimer applies.
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return.[1] And they also emphasize that "Unlike the yield to maturity on a discount bond, the yield to maturity on a coupon bond does not necessarily equal the per-period return if the bond is held to maturity" (p. 401).[2] To date, no analysis of the formal connection between the two concepts for coupon bonds has been provided in the literature, yet.

This paper attempts to study how YTM and rate of return are formally related. It shows rigorously that the YTM for a bondholder exactly measures the theoretic (annual) total rate of return - the current yield plus the rate of capital gain - with the bond price in each year equal to its *amortized book value*. In financial accounting, the amortized book value of an asset for its owner is defined as the present value of the remaining cash inflows from holding the asset discounted at the initial YTM (e.g., see Edmonds *et al*, 2000, pp. 471-477). In practice, such a concept is commonly used in the mortgage market when a borrower may want to pre-pay a mortgage. In that case, the remaining balance is exactly determined by the amortized book value of the loan plus any minor interest accrual between the quote date and actual closing date and transfer (generally electronic) of funds, which is also treated as its "market value". No one has criticized such a practice in the mortgage market.

When the amortized book value of a bond is used to compute its *theoretic* rate of return from holding the bond until maturity, however, we anticipate some possible critiques. This is because there is an institutional difference between the mortgage and the bond markets: in the former the amortized book value of a mortgage loan equals its market value, whereas in the latter the amortized book value of a bond may not be the same as its market price. Regardless of such an institutional difference between the two markets, however, the amortized book value of an asset is always well defined. Based on the amortized book value, we can nevertheless define, and hence, understand, the corresponding theoretic rate of return of any asset for its holder.[3]

2. Definition: Yield to Maturity and Total Rate of Return

The (annual) total rate of return of a coupon bond includes interest payment and capital gain expressed in percentage relative to its initial market price. Formally, let P_t be the market value of a bond at the end of period t, and C be the coupon payment during period t, t = 1, 2,…, N. Then, the total rate of return during period t, r_t, is defined as follows:

$$r_t = \frac{C}{P_{t-1}} + \frac{P_t - P_{t-1}}{P_{t-1}}, \qquad t = 1, 2, \dots, N. \qquad (1)$$

[1] By gross YTM, it means 1 + YTM. Similarly, it is also so defined for gross rate of return.

[2] For coupon bonds, Campbell e*t al* (1997, p. 401-402) have only looked at two important special cases: when the bond is sold at par, i.e., P_0 = F, and when the bond is a consol with infinite term of maturity. In the first case, the YTM equals the coupon rate, C/F, whereas in the second case, the YTM equals the current yield, C/P_t.

[3] Suppose that a bondholder can and must redeem a bond to the bond issuer. Then, the *theoretic* rate of return would become the market one as well, as it happens in the mortgage market.

Note that the total rate of return is an *ex post* measure; it shows explicitly the two sources of earnings – current yield and capital gain. Hence, it is very intuitive and accessible.

On the other hand, if one purchases a coupon bond $\{N, F, C; P_0\}$ - with term to maturity N, face value F, and coupon payment C per period at a price of P_0, its YTM is defined as the unique solution for y (greater than –1) to the following equation: [4, 5]

$$P_0 = \sum_{t=1}^{N} \frac{C}{(1+y)^t} + \frac{F}{(1+y)^N}. \tag{2}$$

Unlike the total rate of return as given in (1), the YTM is implicitly determined from equation (2) and it usually cannot have an explicit solution except in a few special cases. Since the right-hand side of equation (2) is the present value of cash in-flows from holding the bond discounted at y, the YTM is literally interpreted as "the discount rate that equates the present value of the asset's returns with its price today" (Hubbard, 2004, p. 67). A person's subjective discount rate represents his or her *desired* rate of return (Laibson, 2004). Hence, if one pays P_0 to buy a coupon bond $\{N, F, C\}$, then it reveals the rate of return that bond buyer desires to earn is no greater than the YTM as determined in equation (2). This interpretation is tautological, and it shows no obvious connection to the rate of return as defined in equation (1).

However, "[YTM] is broadly viewed as a measure of the average rate of return that will be earned on a bond if it is bought now and held until maturity" (e.g., Bodie *et al*, 2002. p. 426), and YTM "permits us to compare the rate of return on instruments having different cash flows and different maturities" (e.g., Fabozzi and Modigliani, 2002, p. 361). In fact, some authors even directly define "[A] bond's yield to maturity is the rate of return if the bond is held until maturity," and then point out that YTM can be calculated by equation (2) without discussing why (e.g., see Miller and VanHoose, 2004, pp. 82-84). No matter how widely it is believed to be true, the "equivalence" between the YTM (defined as the solution for y in (2)) and the rate of return (r_t as given in (1)) remains an assertion unless it is formally proven.

3. Connection between YTM and Rate of Return

Let's first examine a one-year coupon bond. When N = 1 and $P_1 = F$, we have r as follows

$$r = \frac{C}{P_0} + \frac{F - P_0}{P_0}. \tag{3}$$

Solving for P_0, we have

[4] For the uniqueness of solution to such an equation, see Theorem 6.2(d) on Descartes' Rule of sign, in Henrici (1974, p. 442), for example.

[5] A discount (a.k.a. zero-coupon) bond corresponds to C = 0, whereas a fixed-payment instrument (i.e., a mortgage loan) has F = 0; both are special cases of coupon bond.

$$P_0 = \frac{C + F}{1+r}. \tag{4}$$

Alternatively, we can also derive (3) from (4) by solving for r. It follows that r = y, if N = 1. Thus, we have obtained

Proposition 1. For a one-year coupon bond, the YTM exactly equals the total rate of return.

We now consider an N-year coupon bond. For this purpose, we rewrite (1) by solving for P_{t-1} as follows:

$$P_{t-1} = \frac{C+P_t}{1+r_t}, \qquad \text{for } t = 1, 2, \ldots, N. \tag{5}$$

By recursively substituting P_t into the expression of P_{t-1} in equation (5), we write P_0 as:

$$P_0 = \frac{C+P_1}{1+r_1} = \frac{1}{1+r_1}[C+\frac{1}{1+r_2}(C+P_2)] = \ldots\ldots$$

$$= \sum_{t=1}^{N}\frac{C}{\Pi_{s=1}^{t}(1+r_s)} + \frac{F}{\Pi_{s=1}^{N}(1+r_s)}. \tag{6}$$

Note that $F = P_N$. If $r_1 = r_2 = \ldots = r_N \equiv r > -1$, then equation (6) becomes

$$P_0 = \sum_{t=1}^{N}\frac{C}{(1+r)^t} + \frac{F}{(1+r)^N}. \tag{6a}$$

It follows from (2) and (6a) that r = y, since the solution is unique. That is, if the rate of return remains a constant until maturity, the YTM exactly equals that constant.

Proposition 2. For an N-year coupon bond, the YTM is equal to the annual rate of return if the latter is unchanged over the holding period until maturity.

The market rate of return from holding a bond during period t, r_t, depends on the market prices P_t and P_{t-1}. For all r_t's to be constant, P_t (t = 0, 1, …, N) must satisfy a certain condition. What condition would warrant a constant rate of return? First, we seek for the necessary condition. If $r_1 = r_2 = \ldots = r_N = r$, it follows from the above discussion that r = y, where y is the YTM at time 0 when the bond is purchased at price P_0. It implies from (5) that

$$P_t = \frac{C+P_{t+1}}{1+y} = \frac{1}{1+y}[C + \frac{C+P_{t+2}}{1+y}] = \ldots\ldots$$

$$= \sum_{s=1}^{N-t}\frac{C}{(1+y)^s} + \frac{F}{(1+y)^{N-t}}, \qquad \text{for } t = 1, 2, \ldots, N. \tag{7}$$

Hence, for the rate of return to be constant, the bond price must always equal the present value of the remaining cash in-flows of the investment discounted at the initial YTM.

It can be shown that equation (7) is also a sufficient condition for a constant rate of return. In particular, applying (7) to time (t – 1), we have

$$P_{t-1} = \sum_{s=1}^{N-t+1} \frac{C}{(1+y)^s} + \frac{F}{(1+y)^{N-t+1}}$$

$$= \frac{1}{1+y} \left[C + \sum_{s=1}^{N-t} \frac{C}{(1+y)^s} + \frac{F}{(1+y)^{N-t}} \right]$$

$$= \frac{C+P_t}{1+y}, \qquad \text{for } t = 1, 2, \ldots, N. \qquad (8)$$

It follows from (5) and (8) that

$$r_t = y, \qquad \text{for } t = 1, 2, \ldots, N.$$

Therefore, the sufficient and necessary condition for a constant rate of return is that the bond price always satisfies equation (7). We summarize the above discussion in the following

Proposition 3. The rate of return remains a constant every year if and only if the bond price always equals to the present value of the remaining cash in-flows of the bond discounted at the initial YTM.

In financial accounting, when an asset is valuated by the present value of its remaining cash in-flows discounted at the initial interest rate, it is referred to as its *amortized book value* (e.g., Edmonds *et al*, 2000, pp. 471-477). Thus, Proposition 3 essentially indicates that the (initial) YTM of a bond equals the theoretic total rate of return, which is obtained with the bond valuated by its amortized book value.

Corollary 1. The YTM of a bond {F, C, N; P_0} as determined by the solution for y in equation (2) measures the theoretic rate of return with the bond being valuated by its amortized book value in each period.

Example. We provide a numerical example to illustrate the above results. Suppose that F = $1,000, C = $40, N = 5, and P_0 = $956.71. Then, we have YTM = 5.0%. Valuated at such an YTM, the amortized book values of the bond from year 1 to year 5 are: P_1 = $964.54, P_2 = $972.77, P_3 = $981.41, P_4 = $990.48 and P_5 = F = $1,000. We can verify that the corresponding theoretic total rate of return at year t,

$$r_t = (P_t - P_{t-1} + C)/P_{t-1} = 5.0\%, \qquad \text{for } t = 1, 2, \ldots, 5.$$

Note that this theoretic rate of return may not equal the market rate of return, which is based on the market bond price. It is because the bond price changes constantly in response to the economic and financial conditions and hence it may not equal its amortized book value. Naturally, a critique might be: How relevant is this theoretic rate of return in theory and in practice? We provide three explanations.

First, the concept of amortized book value is actually employed in the real world. For example, when a borrower wants to pay off a mortgage loan *prior to* its maturity, the remaining balance is exactly determined by its current amortized book value. No one has criticized such a practice commonly employed in the mortgage market when the YTM (referred to as *the effective rate of return* in a mortgage contract) is used to determine the remaining balance. Why? It is because what the *borrower* pays back to the *lender* is exactly the amortized book value, whereas a bondholder resells the bond in the secondary bond market at the *current* market price, which may not be same as its current amortized book value. This institutional difference between the two markets may make it a bit harder for people to accept the concept of amortized book value for a bond than that for a mortgage loan. No matter whether there is a real-world counterpart, the theoretic rate of return is well defined and accessible for a bond as well as for a mortgage loan. The YTM exactly equals this theoretic rate of return based on amortized book value.

Second, in theory it is a self-fulfilling property with a fixed-point feature. When a bond buyer pays P_0 for a bond {F, C, N}, it reveals that her *desired* rate of return is no greater than the YTM as determined in equation (2). If she maintains such a desired rate of return and continues valuating the bond every year by its amortized book value, then she will indeed, in psychology as well as in accounting, earn such an (annual) rate of return as measured by the YTM every year until maturity.

Third, the connection between the theoretic rate of return and the market rate of return is similar to that between the long-run equilibrium and the short-run equilibrium in macroeconomics. For example, the current unemployment rate fluctuates around the natural rate of unemployment, and the actual GDP fluctuates around the potential GDP. For a bond, its current market price fluctuates around its amortized book value, in particular, when it is close to the maturity. Therefore, when one claims that the YTM of a coupon bond measures the average (annual) rate of return from holding it until maturity, it is an *ex ante* and theoretic statement rather than an *ex post* or empirical statement, just like how the natural rate of unemployment measures the average rate of actual unemployment.

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